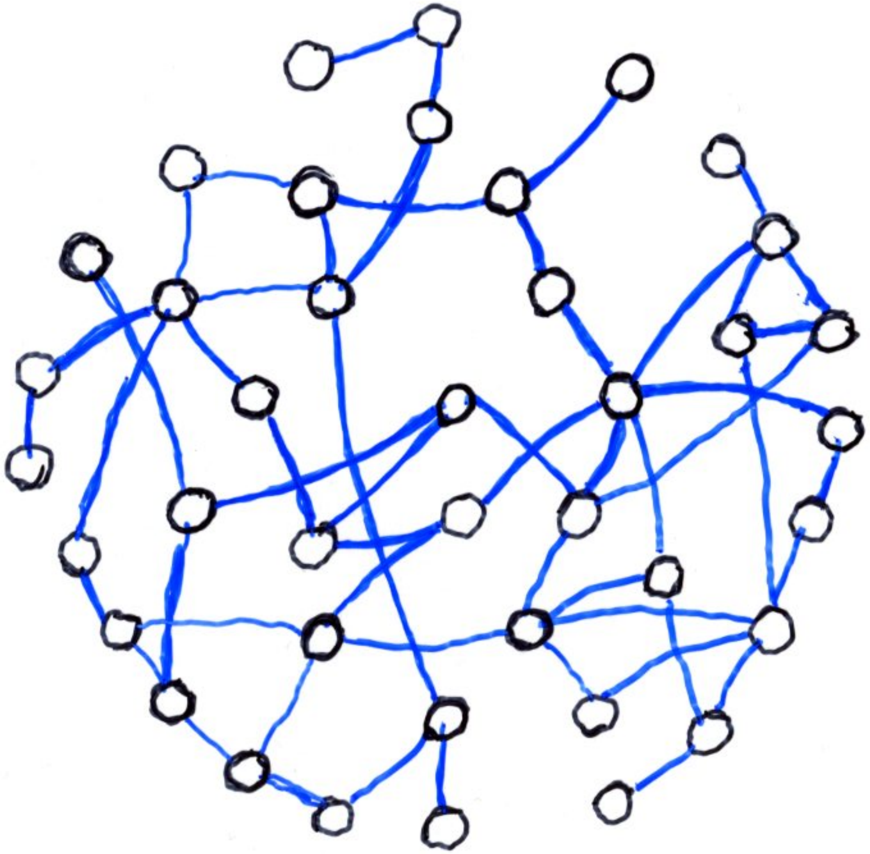
ON THE APPROXIMABILITY OF INDEPENDENT SET PROBLEM IN POWER LAW GRAPHS

MATHIAS HAUPTMANN

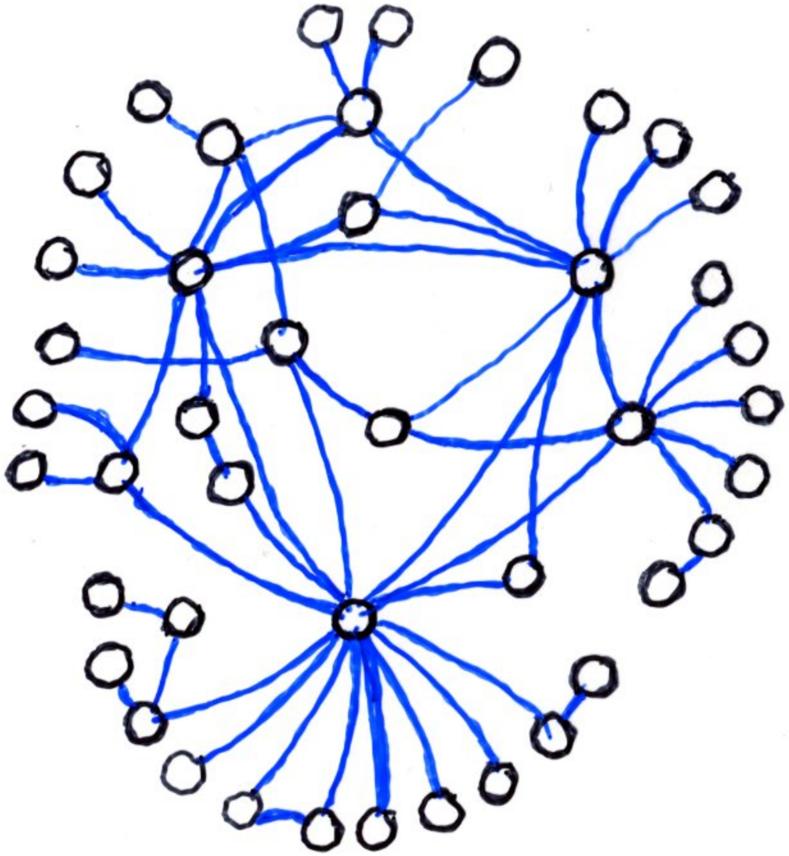
JOINT WORK WITH MAREK KARPINSKI

POWER LAW GRAPHS

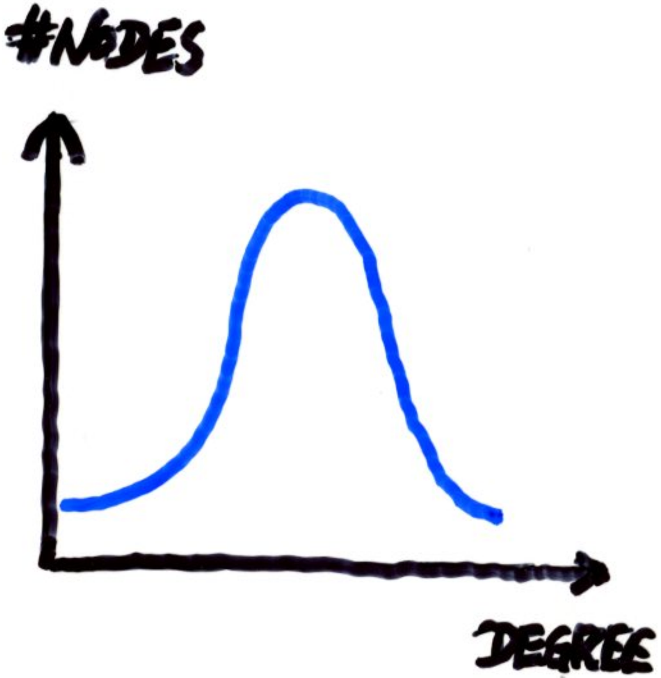
ERDÖS - RÉNYI



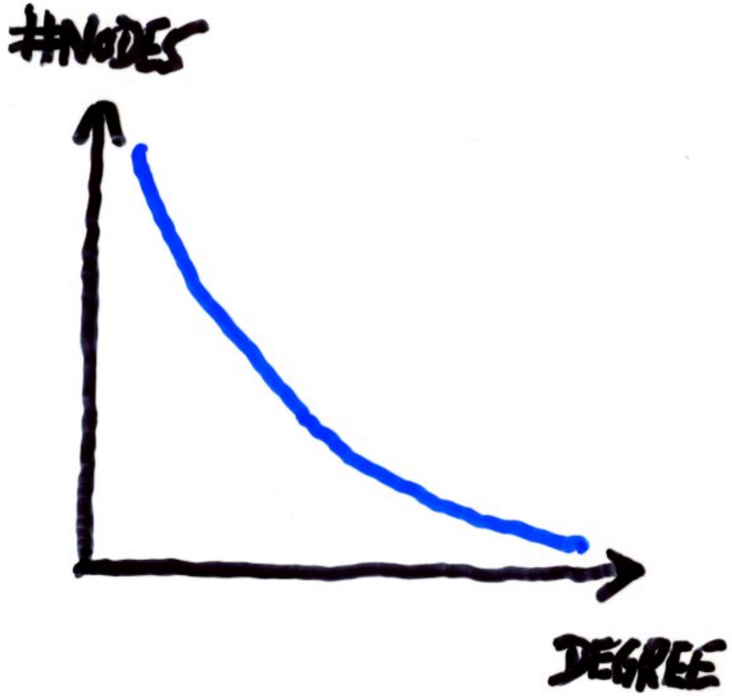
POWER LAW GRAPH (PLG)



POWER LAW GRAPHS



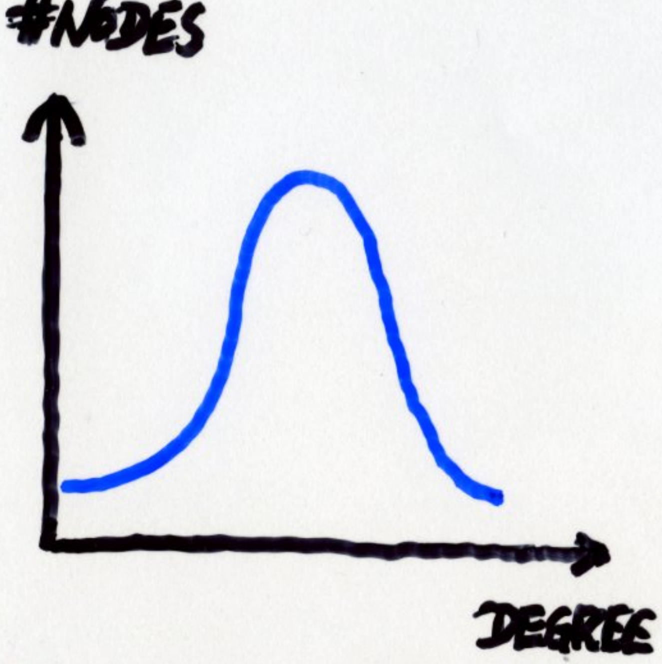
ERDÖS -RÉNYI



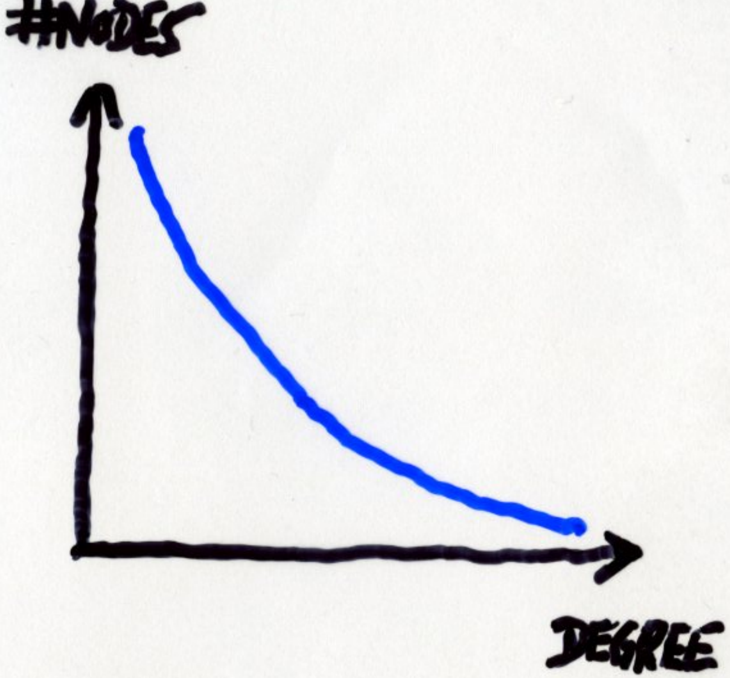
POWER LAW

(SCALE - FREE)

POWER LAW GRAPHS



ERDÖS-RÉNYI

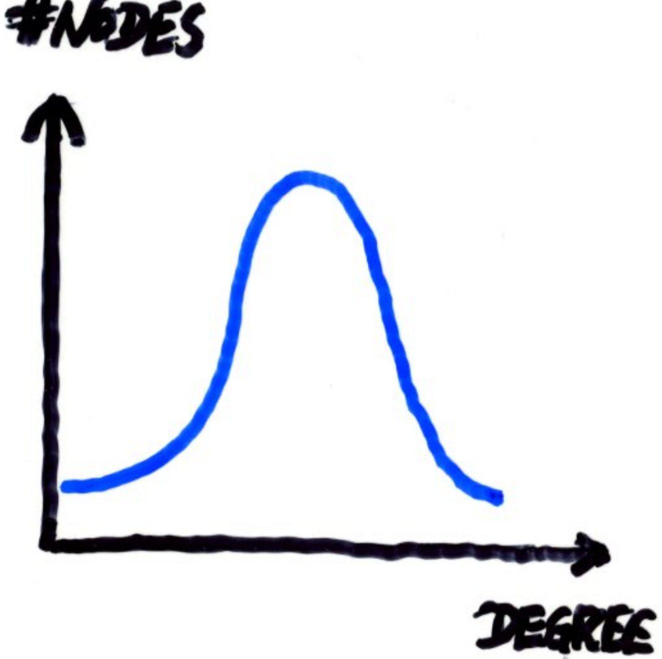


POWER LAW

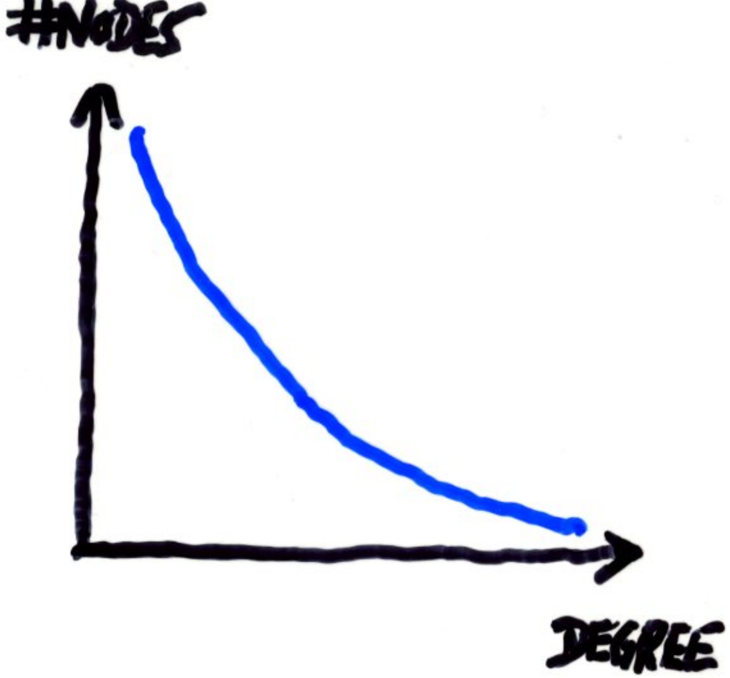
(SCALE-FREE)

$$P(k) \sim k^{-\beta}$$

POWER LAW GRAPHS



ERDÖS-RÉNYI



POWER LAW

(SCALE-FREE)

β = POWER LAW EXPONENT → $P(k) \sim k^{-\beta}$

OUTLINE

- POWER LAW GRAPHS (PLG)
- MAXIMUM INDEPENDENT SET (MIS)
- NEW APPROXIMATION LOWER BOUNDS FOR MIS ON POWER LAW GRAPHS
  - AUXILIARY GRAPH CONSTRUCTION
  - CASE $\beta < 1$
  - CASE $\beta = 1$
- OPEN PROBLEMS

EXAMPLES OF SCALE-FREE NETWORKS

- COMPUTER NETWORKS
  EG. WEBGRAPH OF THE WWW
- SOCIAL NETWORKS, COLLABORATION NETWORKS
- FINANCIAL NETWORKS
  EG. INTERBANK PAYMENT NETWORKS
- BIOLOGICAL NETWORKS
  - PROTEIN INTERACTION NETWORKS

(α, β)-POWER LAW GRAPHS

MAX DEGREE $\Delta = \lfloor e^{\alpha/\beta} \rfloor$

NODES OF DEGREE i: $y_i = \left\lfloor \frac{e^\alpha}{i^\beta} \right\rfloor$

(α, β)-POWER LAW GRAPHS

MAX DEGREE $\quad \Delta = \lfloor e^{\alpha/\beta} \rfloor$

NODES OF DEGREE i: $\quad y_i = \left\lfloor \frac{e^{\alpha}}{i^{\beta}} \right\rfloor$

# NODES	FOR
$\frac{e^{\alpha/\beta}}{1-\beta}$	$0<\beta<1$
$\alpha \cdot e^{\alpha}$	$\beta=1$
$\zeta(\beta) \cdot e^{\alpha}$	$\beta>1$

$$n \approx \begin{cases} \frac{e^{\alpha/\beta}}{1-\beta} & 0<\beta<1 \\ \alpha \cdot e^{\alpha} & \beta=1 \\ \zeta(\beta) \cdot e^{\alpha} & \beta>1 \end{cases}$$

(α, β)-POWER LAW GRAPHS

MAX DEGREE $\quad \Delta = \lfloor e^{\alpha/\beta} \rfloor$

NODES OF DEGREE i: $\quad y_i = \left\lfloor \frac{e^{\alpha}}{i^{\beta}} \right\rfloor$

#NODES	FOR	#EDGES	FOR
$n \approx \frac{e^{\alpha/\beta}}{1-\beta}$	$0<\beta<1$	$\frac{1}{2}\,\frac{e^{2\alpha/\beta}}{2-\beta}$	$0<\beta<2$
$n \approx \alpha \cdot e^{\alpha}$	$\beta=1$	$\frac{1}{4}\,\alpha e^{\alpha}$	$\beta=2$
$n \approx \zeta(\beta) \cdot e^{\alpha}$	$\beta>1$	$\frac{1}{2}\,\zeta(\beta-1) e^{\alpha}$	$\beta>2$

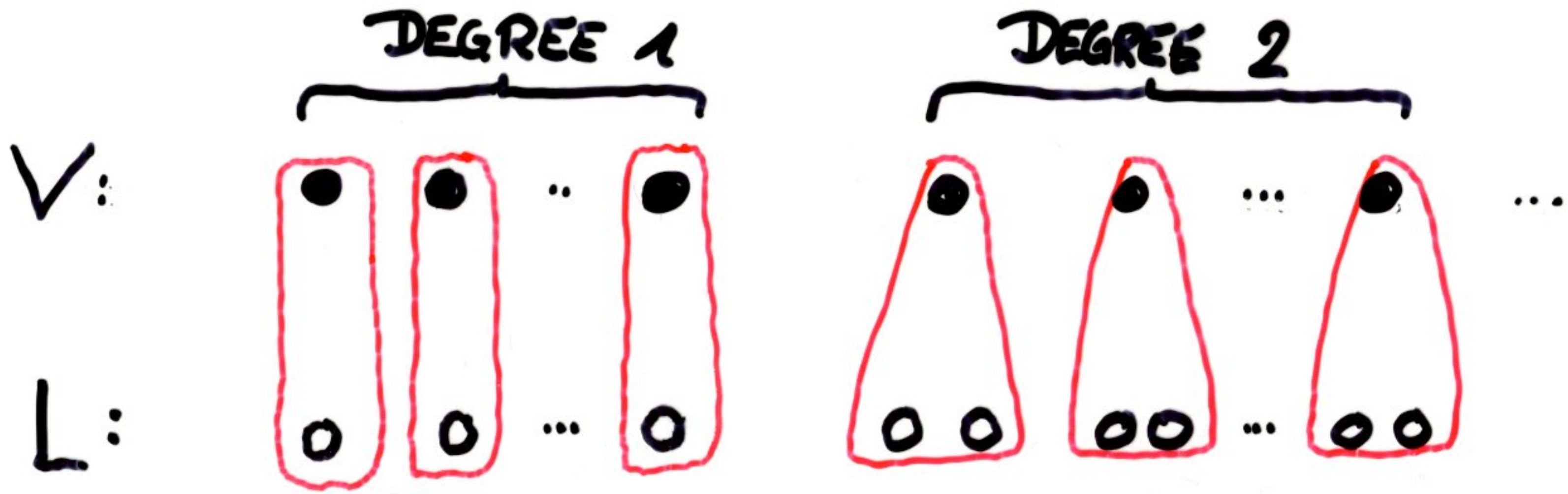
RANDOM PLGs [ACL01]

1. GENERATE SET L OF $d(v)$ COPIES FOR EACH $v \in V$
2. M := RANDOM PERFECT MATCHING ON L
3. M INDUCES (MULTI-)GRAPH G

V:

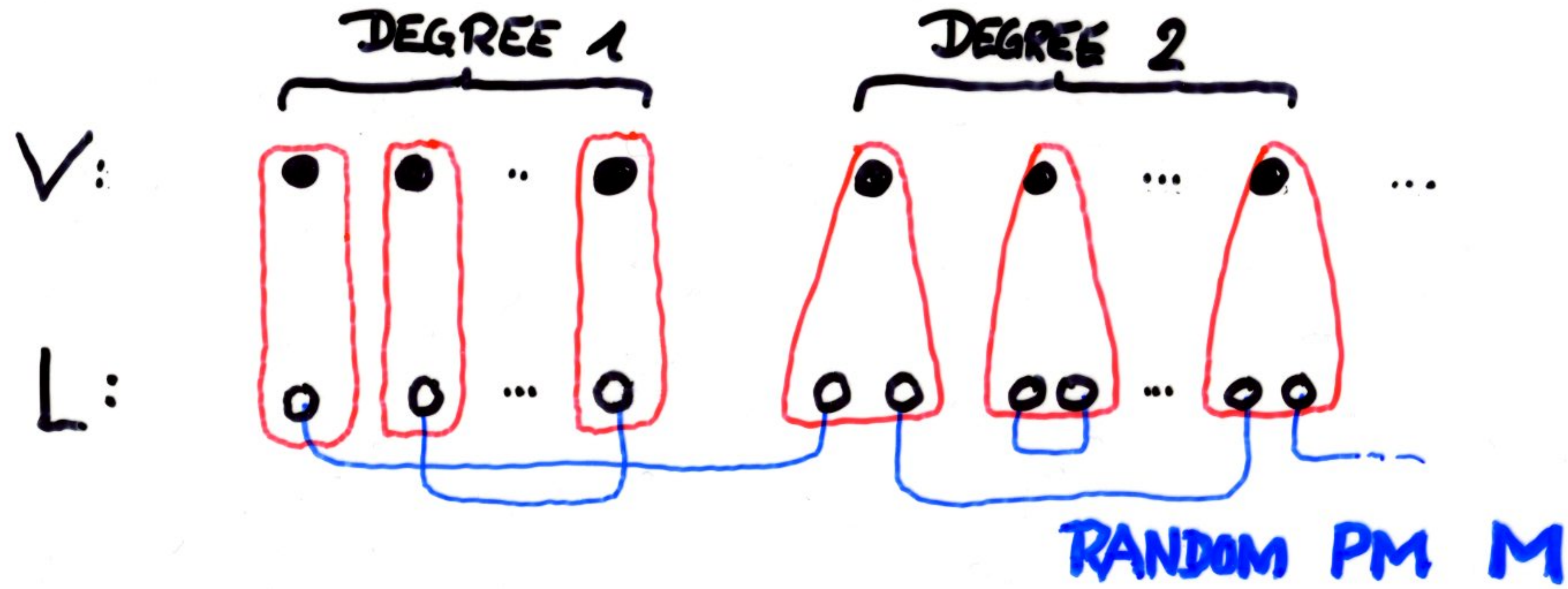
RANDOM PLGs [ACL01]

1. GENERATE SET L OF $d(v)$ COPIES FOR EACH $v \in V$
2. M := RANDOM PERFECT MATCHING ON L
3. M INDUCES (MULTI-)GRAPH G



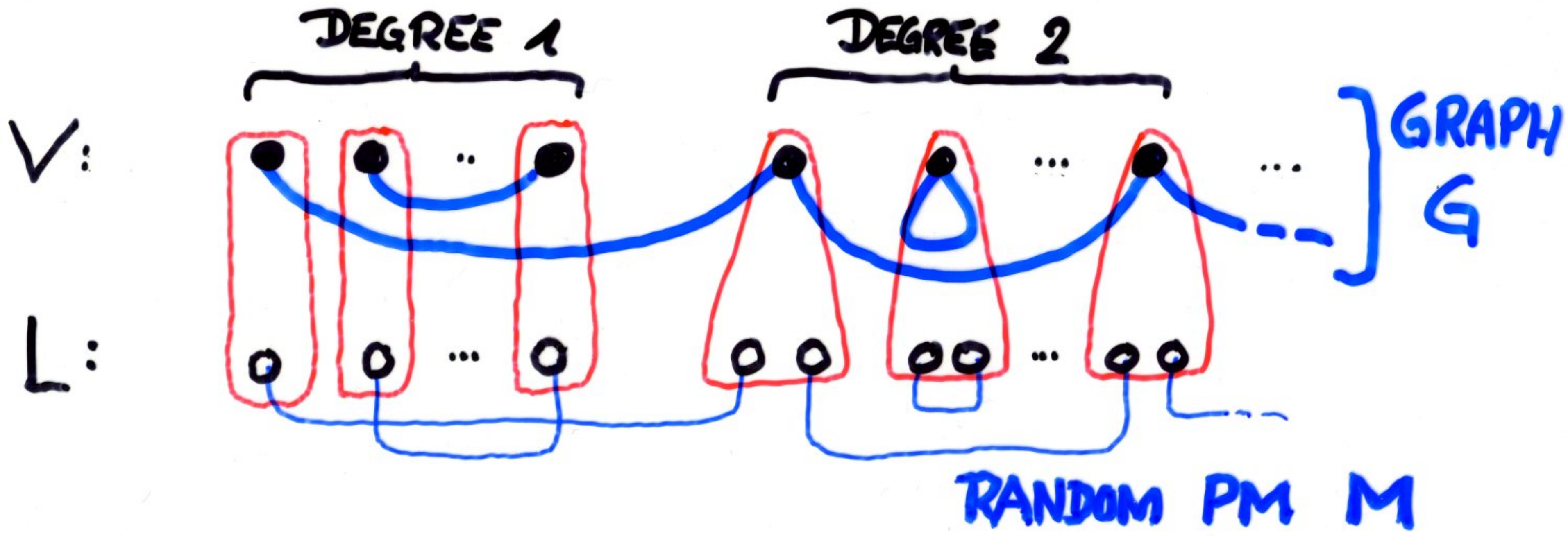
RANDOM PLGs [ACL01]

1. GENERATE SET L OF $d(v)$ COPIES FOR EACH $v \in V$
2. M := RANDOM PERFECT MATCHING ON L
3. M INDUCES (MULTI-)GRAPH G

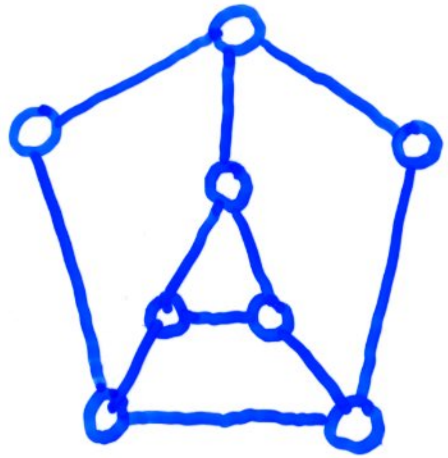


RANDOM PLGs [ACL01]

1. GENERATE SET L OF $d(v)$ COPIES FOR EACH $v \in V$
2. M := RANDOM PERFECT MATCHING ON L
3. M INDUCES (MULTI-)GRAPH G

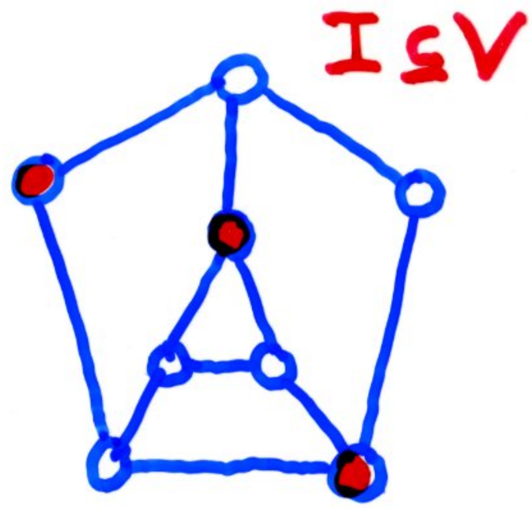


MAXIMUM INDEPENDENT SET (MIS)



GRAPH G

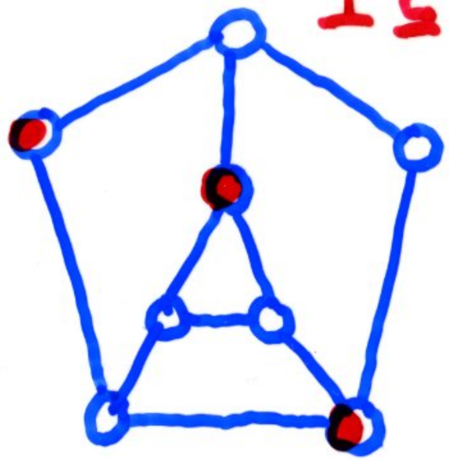
MAXIMUM INDEPENDENT SET (MIS)



GRAPH G

MAXIMUM INDEPENDENT SET (MIS)

$I \subseteq V$



GRAPH G

STATUS OF MIS

- MIS EQUIVALENT TO MAX-CLIQUE
- NP-HARD TO APPROX. WITHIN $n^{1-\varepsilon}$ (FOR EVERY $\varepsilon > 0$) [Z07]
- NOT APPROXIMABLE WITHIN $n / 2^{(\log n)^{1-\gamma}}$ UNLESS $\tilde{NP} = \tilde{P}$

MIS IN COMPLEX NETWORKS

- SCHEDULING IN WIRELESS SENSOR NETWORKS
  EFFICIENTLY EMPTYING WSN
- SECONDARY STRUCTURE MOTIFS IN PROTEINS
- COMMUNITY STRUCTURE IN COMPLEX NETWORKS
- EPIDEMICS

MIS IN POWER LAW GRAPHS

PREVIOUS RESULTS

- NP-HARD FOR ALL $\beta > 0$ (Ferrante et al. '08)
- APX-HARD FOR ALL $\beta > 1$ (Shen et al. '12)
  EXPLICIT APPROX. LOWER BOUNDS

MIS IN POWER LAW GRAPHS

PREVIOUS RESULTS

- NP-HARD FOR ALL $\beta > 0$ (Ferrante et al. '08)
- APX-HARD FOR ALL $\beta > 1$ (Shen et al. '12)
  EXPLICIT APPROX. LOWER BOUNDS

OUR RESULTS

- FOR $\beta < 1$, NP-HARD TO APPROX. WITHIN $n^{1-\beta-\varepsilon}$ FOR ALL $\varepsilon > 0$
- FOR $\beta = 1$, LOWER BOUND $(\log n)^{\varepsilon}$ FOR SOME $\varepsilon \in (0,1)$

THE CASE $\beta < 1$

OUTLINE

THE CASE $\beta < 1$

OUTLINE

$n^{1-\varepsilon}$ LOWER BOUND
FOR MIS (ZUCKERMAN'07)

THE CASE $\beta < 1$

OUTLINE

CLASS OF GRAPHS

$\mathcal{G} = \mathcal{G}_{n^{\varepsilon}} \cup \mathcal{G}_{n^{1-\varepsilon}}$

$n^{1-\varepsilon}$ LOWER BOUND
FOR MIS (ZUCKERMAN '07)

THE CASE $\beta < 1$

OUTLINE

CLASS OF GRAPHS

$$\boxed{\text{STEP 1}}$$

$$\mathcal{G} = \mathcal{G}_{n^{\varepsilon}} \cup \mathcal{G}_{n^{1-\varepsilon}} \ni G \mapsto G'$$

HAS PERF. MATCHING

$n^{1-\varepsilon}$ LOWER BOUND FOR MIS (ZUCKERMAN '07)

THE CASE $\beta < 1$

OUTLINE

CLASS OF GRAPHS

$$\mathcal{G} = \mathcal{G}_{n^{\varepsilon}} \cup \mathcal{G}_{n^{1-\varepsilon}} \ni G \xmapsto{\boxed{\text{STEP 1}}} G' \xmapsto{\boxed{\text{STEP 2}}} \underbrace{G' \cup A}_{\text{PLG}}$$

G' HAS PERF. MATCHING

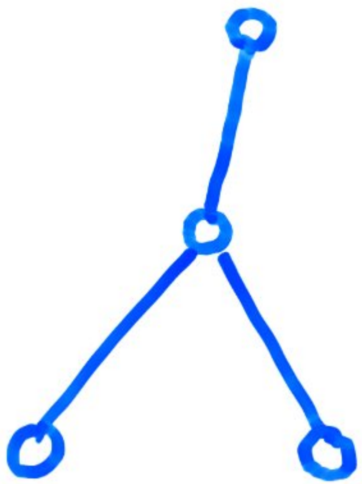
$n^{1-\varepsilon}$ LOWER BOUND

FOR MIS (ZUCKERMAN '07)

EMBEDDING OF GRAPHS INTO POWER LAW GRAPHS

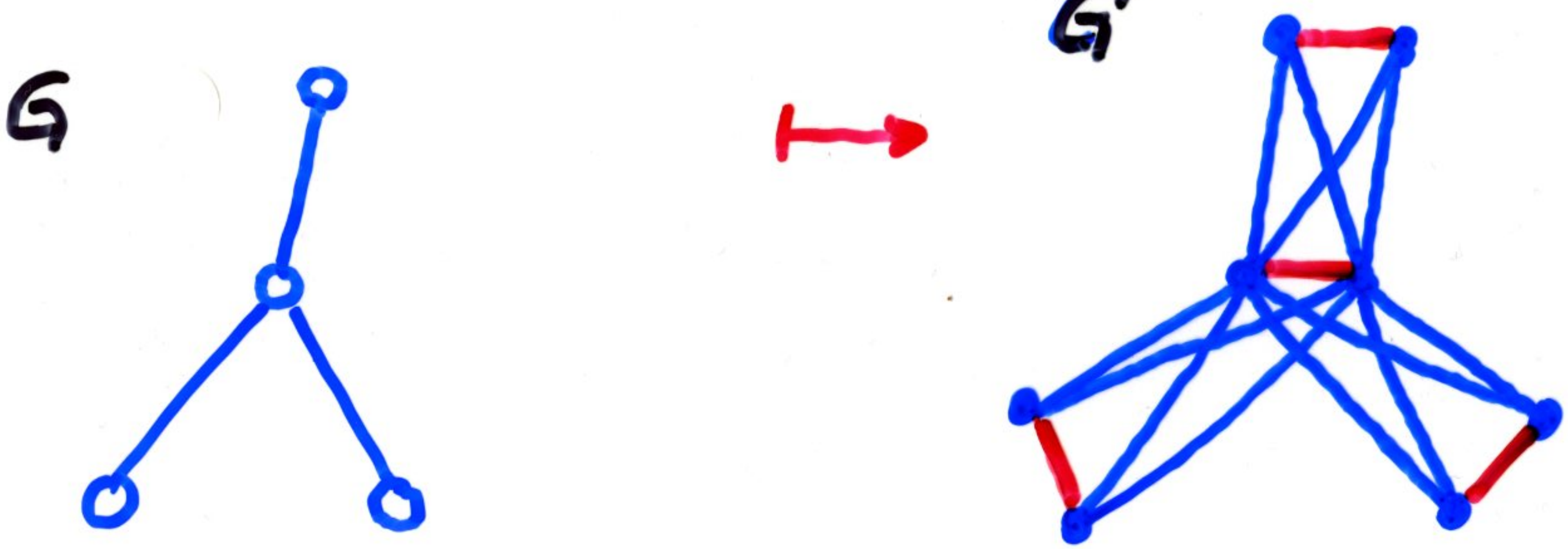
STEP 1 $\quad G \mapsto G'$ SUCH THAT G' HAS PERFECT MATCHING

G



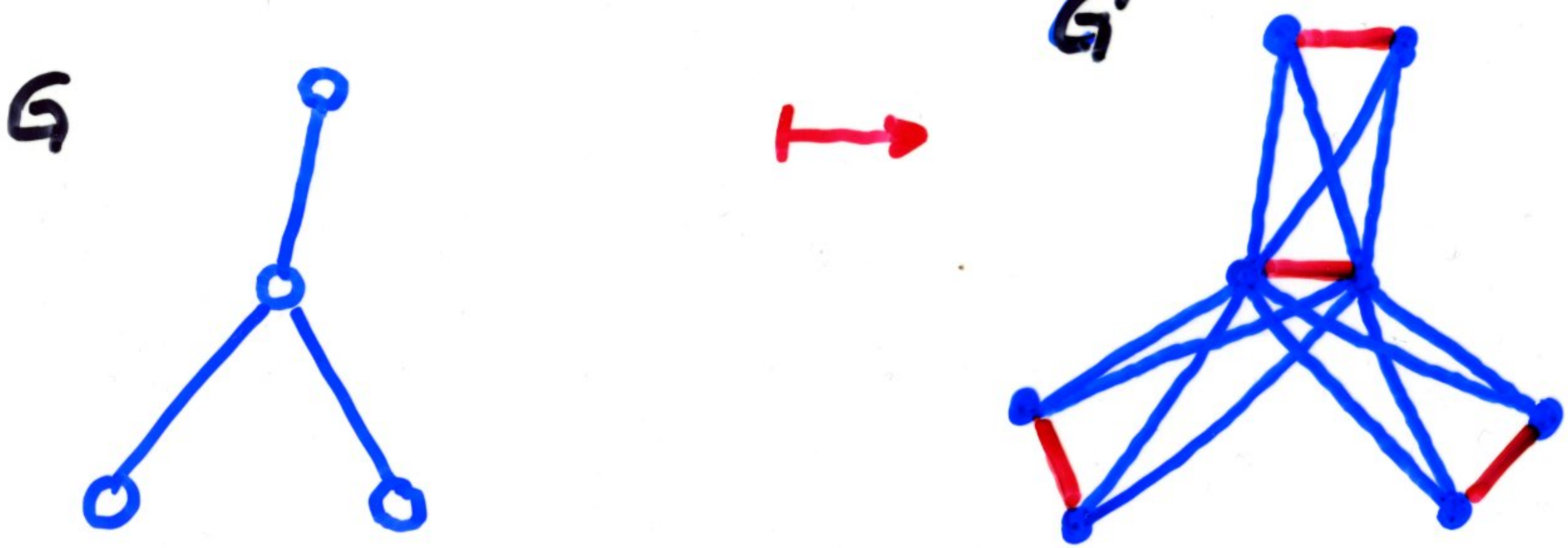
EMBEDDING OF GRAPHS INTO POWER LAW GRAPHS

STEP 1 $G \mapsto G'$ SUCH THAT G' HAS PERFECT MATCHING



EMBEDDING OF GRAPHS INTO POWER LAW GRAPHS

<u>STEP 1</u> $G \mapsto G'$ SUCH THAT G' HAS PERFECT MATCHING



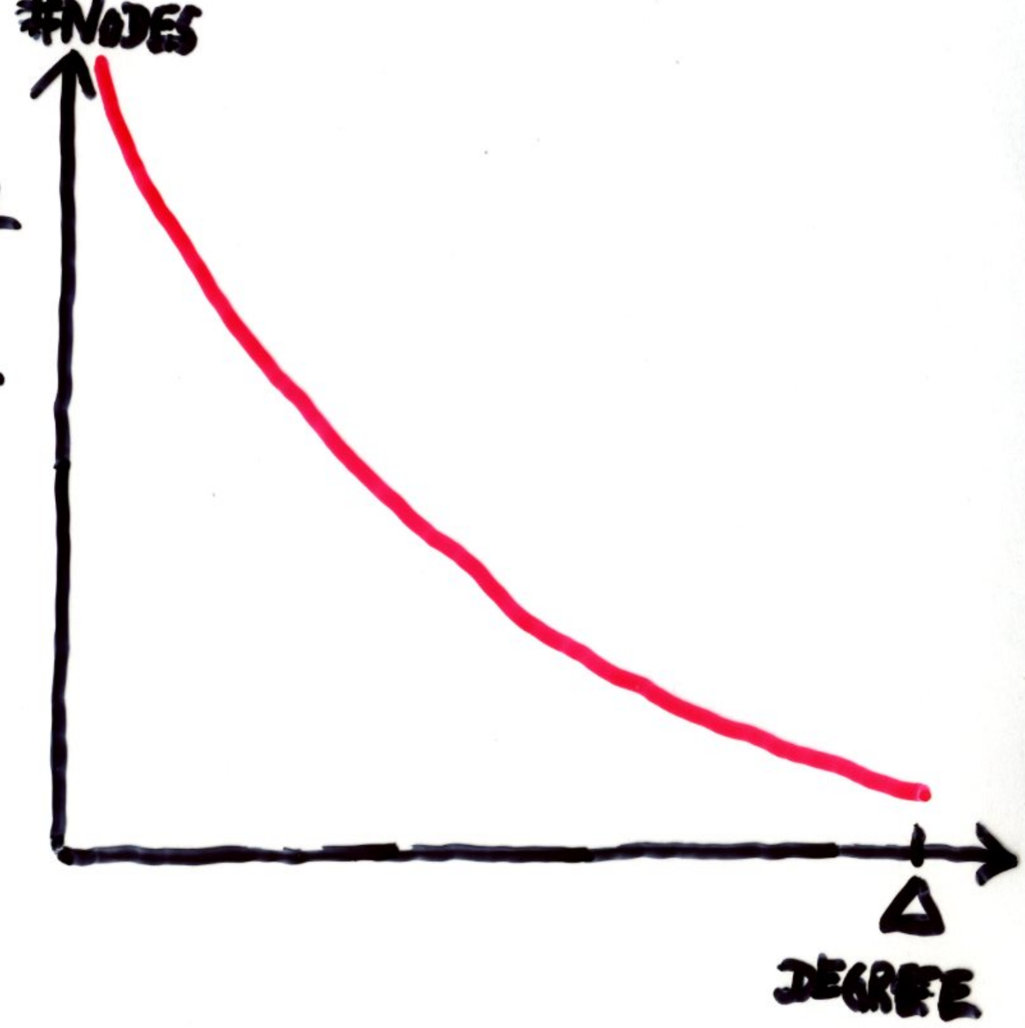
<u>PROPERTY:</u> $IS(G') = IS(G)$

EMBEDDING OF G' INTO PLG
– CASE $\beta < 1$ –

▷ USE PERFECT MATCHING TO FIT G' INTO INTERVAL $[x\Delta, \Delta]$ OF PL DISTRIB.

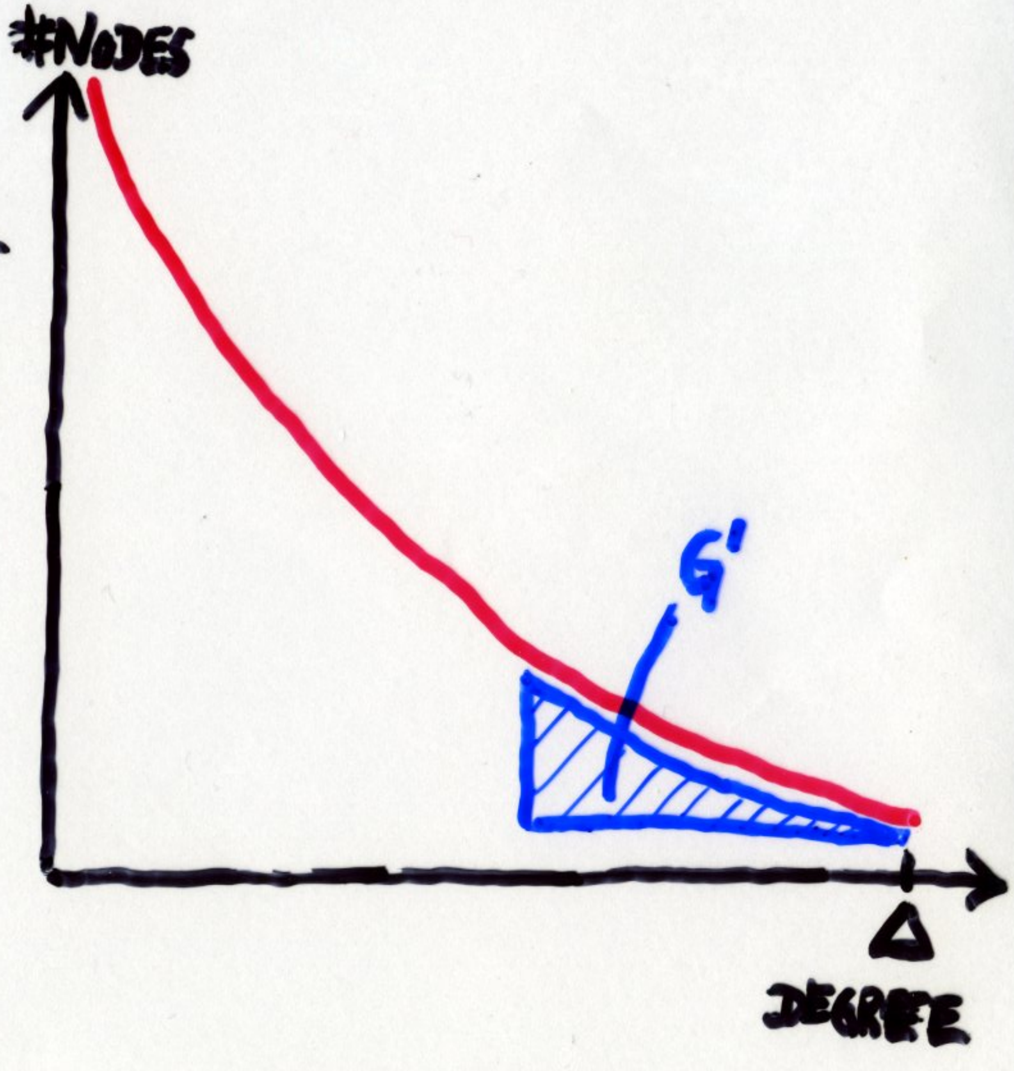
EMBEDDING OF G' INTO PLG
– CASE $\beta < 1$ –

▷ USE PERFECT MATCHING TO FIT G' INTO INTERVAL $[x\Delta, \Delta]$ OF PL DISTRIB.



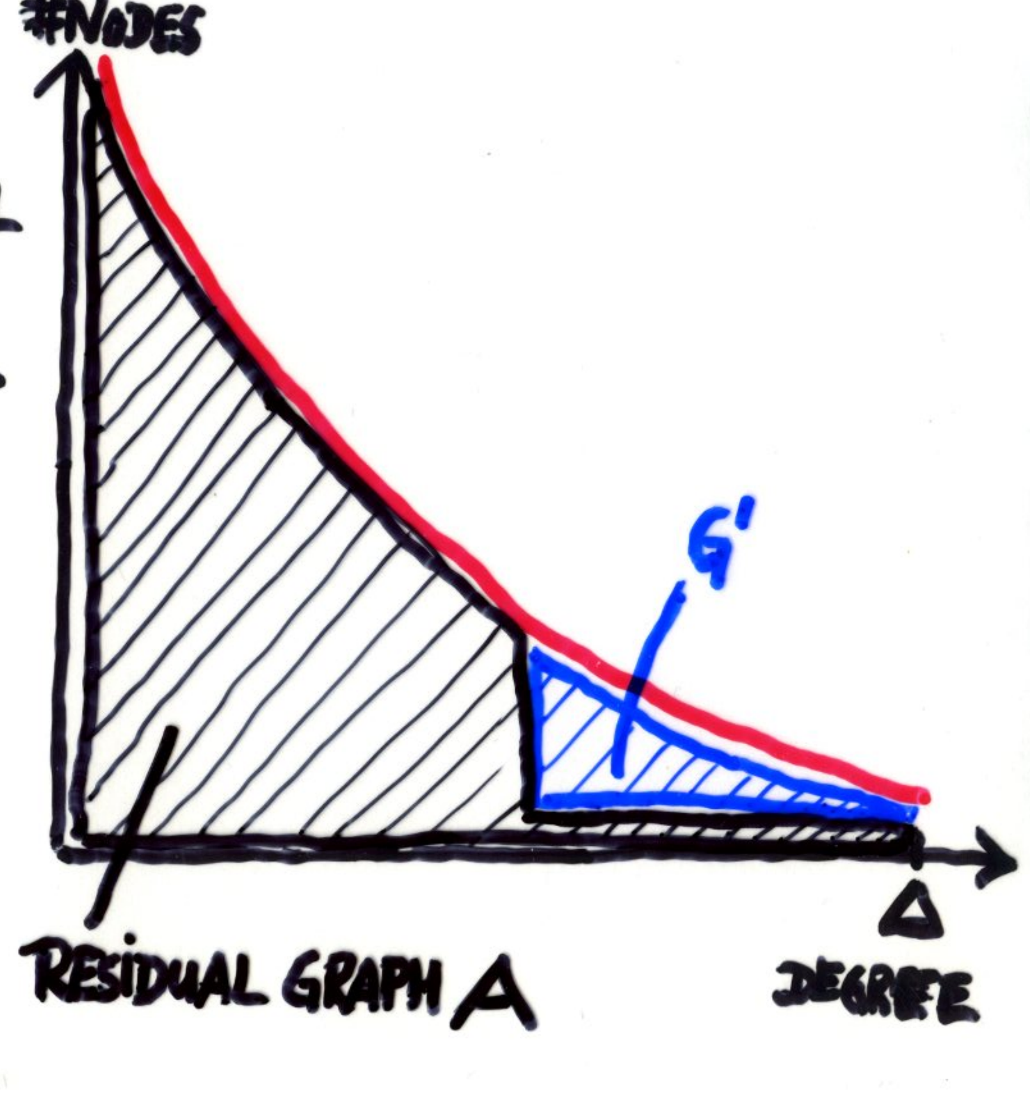
EMBEDDING OF G' INTO PLG
- CASE $\beta < 1$ -

- USE PERFECT MATCHING TO FIT G' INTO INTERVAL $[x\Delta, \Delta]$ OF PL DISTRIB.

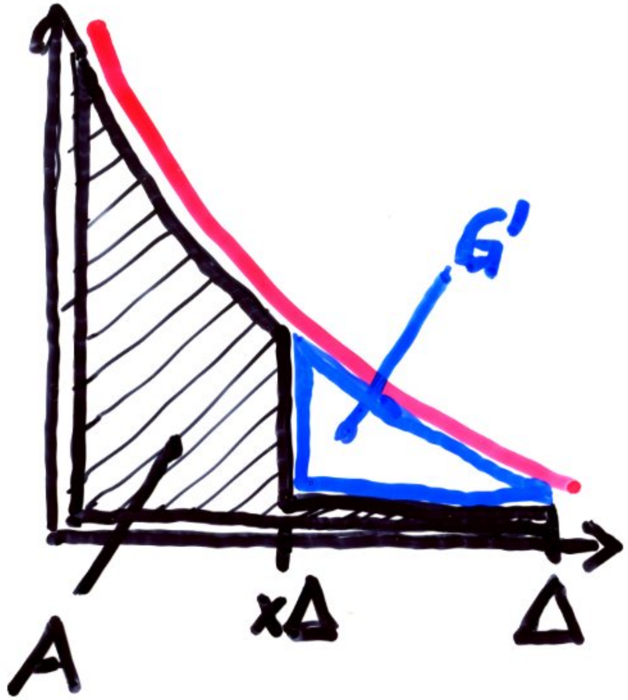


EMBEDDING OF G' INTO PLG
- CASE $\beta < 1$ -

- USE PERFECT MATCHING TO FIT G' INTO INTERVAL $[x\Delta, \Delta]$ OF PL DISTRIB.



CONSTRUCTION OF A



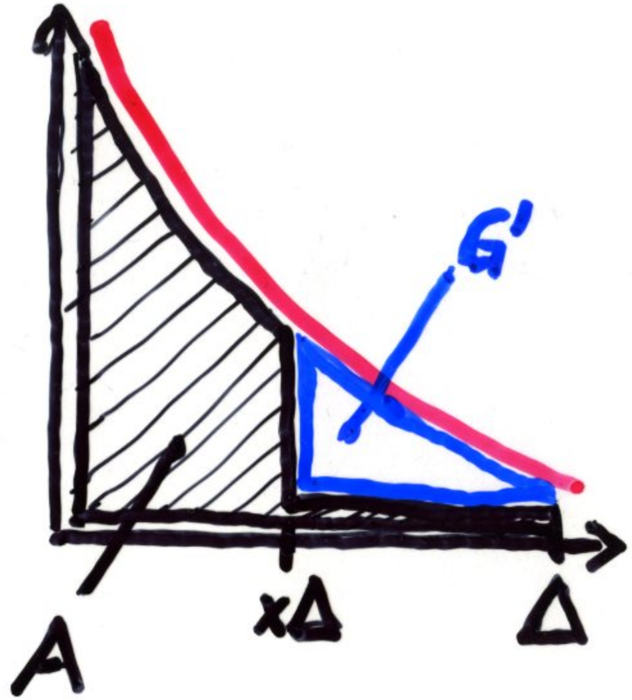
CONSTRUCTION OF A

IN GENERAL

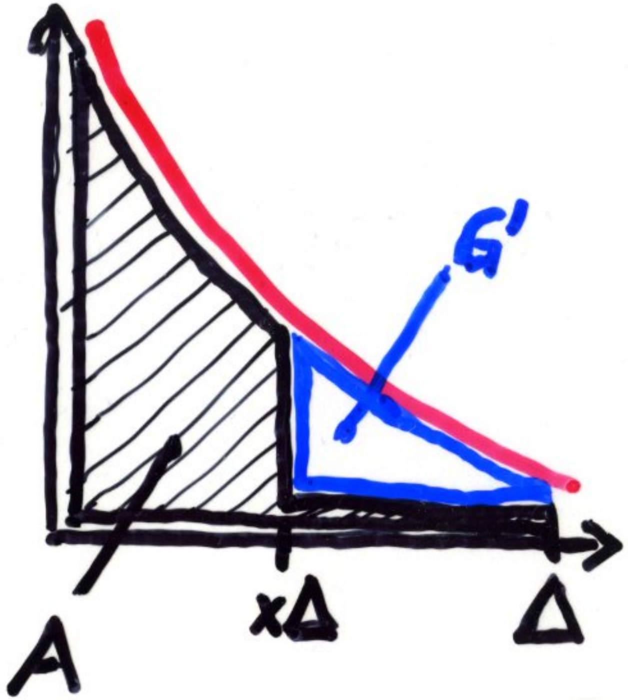
- GIVEN DEGREE SEQUENCE

$$d = (d_1, \dots d_N)$$

- CONSTRUCT MULTIGRAPH A
WITH N NODES, DEG. SEQUENCE d
SUCH THAT $IS(A)$ "is SMALL"



CONSTRUCTION OF A



IN GENERAL

- GIVEN DEGREE SEQUENCE

$$d = (d_1, \dots d_N)$$

- CONSTRUCT MULTIGRAPH A WITH N NODES, DEG. SEQUENCE d SUCH THAT $IS(A)$ "is SMALL"

APPROACH

CLIQUE

$$d_i \leq d_{i+1} \dots \leq d_{i+d_i-1}$$

CONSTRUCTION OF A

- DEGREE SEQUENCE FOR INTERVAL $[a,b]$, $1 \leq a \leq b \leq \Delta$

$$d = (\underbrace{a, \ldots\ldots a}_{\frac{e^{\alpha}}{a^{\beta}}}, a+1, \ldots a+1, \ldots\ldots \underbrace{b, \ldots b}_{\frac{e^{\alpha}}{b^{\beta}}})$$

CONSTRUCTION OF A

- DEGREE SEQUENCE FOR INTERVAL $[a,b]$, $1 \leq a \leq b \leq \Delta$

$$d = (\underbrace{a, \ldots\ldots a}_{\frac{e^\alpha}{a^\beta}}, a+1, \ldots a+1, \ldots\ldots \underbrace{b, \ldots b}_{\frac{e^\alpha}{b^\beta}})$$

- BOUND ON SIZE OF MIS IN A

CONSTRUCTION OF A

- DEGREE SEQUENCE FOR INTERVAL $[a,b]$, $1 \leq a \leq b \leq \Delta$

$$d = (\underbrace{a,\dots\dots a}_{\frac{e^{\alpha}}{a^{\beta}}}, a+1,\dots a+1, \dots\dots \underbrace{b,\dots b}_{\frac{e^{\alpha}}{b^{\beta}}})$$

- BOUND ON SIZE OF MIS IN A

PARTITION OF $[a,b]$ INTO SUBINTERVALS

ESTIMATE FOR IS(A)

ESTIMATE FOR IS(A)

$$IS(A) = IS(G_{[a,b]}) \leq \sum_{j=0}^{k-1} \left\lceil \frac{\displaystyle\sum_{\ell=a_j}^{a_{j+1}-1} \left\lfloor \frac{e^{\alpha}}{\ell^{\beta}} \right\rfloor}{a_j} \right\rceil$$

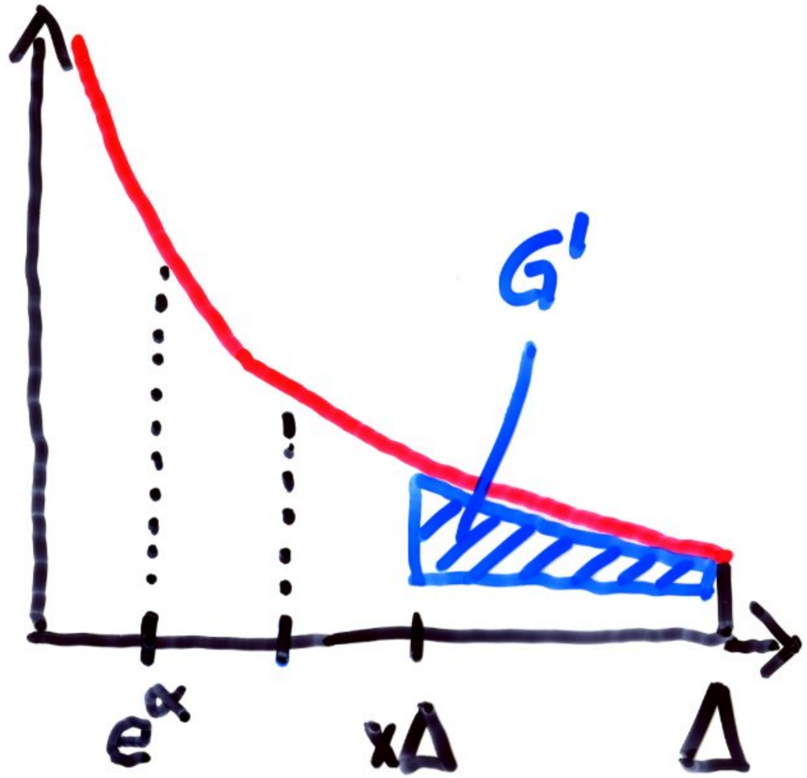
ESTIMATE FOR IS(A)

$$IS(A) = IS(G_{[a,b]}) \leq \sum_{j=0}^{k-1} \left\lceil \frac{\sum_{\ell = a_j}^{a_{j+1}-1} \left\lfloor \frac{e^\alpha}{\ell^\beta} \right\rfloor}{a_j} \right\rceil$$

- FOR $\beta < 1$: $k = \text{CONST}$
- FOR $\beta = 1$: $k = \log n$

THE CASE $\beta < 1$ CONT.

- SPLIT $[1, \Delta]$ INTO FOUR PARTS

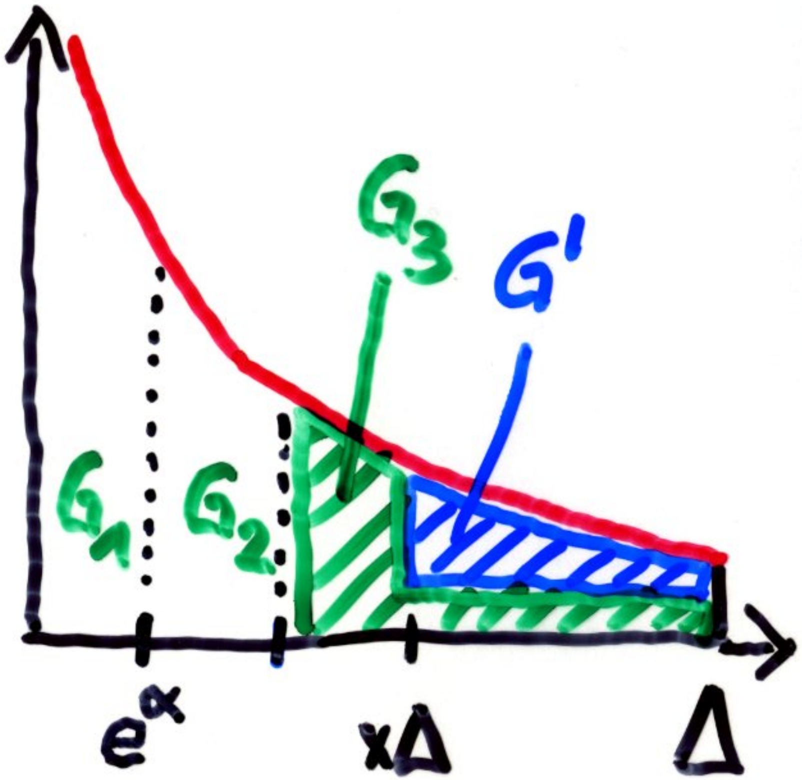


THE CASE $\beta < 1$ CONT.

- SPLIT $[1, \Delta]$ INTO FOUR PARTS

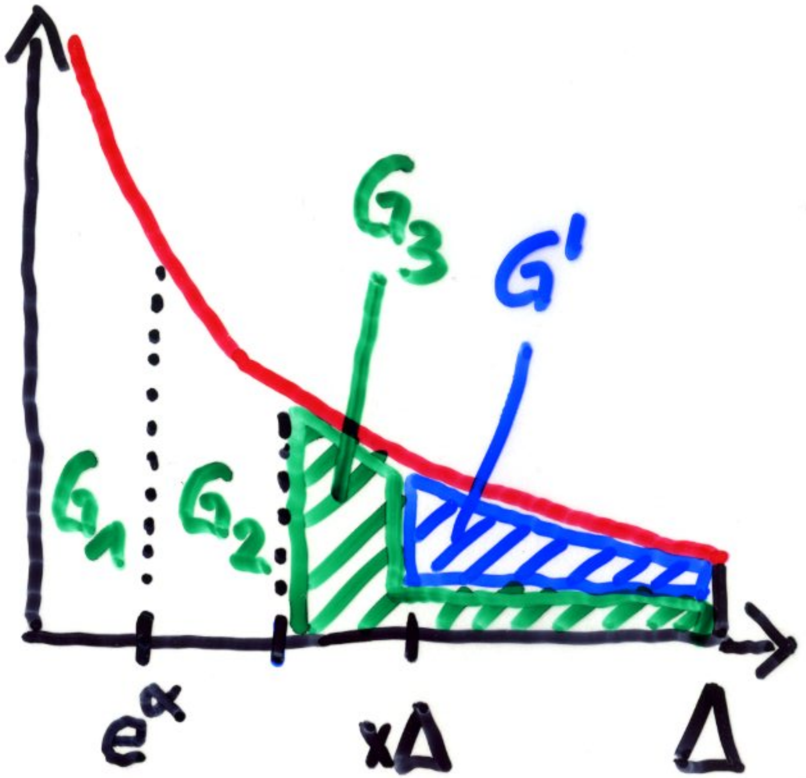
- FOR $i = 1, 2, 3$ SHOW THAT

  $IS(G_i) \leq \text{CONST} \cdot e^{\alpha}$



THE CASE $\beta < 1$ CONT.

- SPLIT $[1,\Delta]$ INTO FOUR PARTS
- FOR $i = 1,2,3$ SHOW THAT $IS(G_i) \le \text{CONST} \cdot e^{\alpha}$



<u>THEOREM</u>

FOR $\beta < 1$, MIS ON (α,β)-PLG IS NP-HARD TO APPROXIMATE WITHIN $n^{1-\beta-\varepsilon}$

$\boxed{\text{THE CASE } \beta = 1}$

WE HAVE

$$\#\text{NODES} = \alpha e^{\alpha}$$

e^{α} NODES OF DEGREE 1

THE CASE $\beta = 1$

WE HAVE

$$\# \text{NODES} = \alpha e^{\alpha}$$

e^{α} NODES OF DEGREE 1

$\Rightarrow$ $O(\log n)$-APPROX. FOR MIS in (α, β)-PLG FOR $\beta = 1$

THE CASE $\beta = 1$

WE HAVE

$$\#\text{NODES} = \alpha e^{\alpha}$$

e^{α} NODES OF DEGREE 1

$\Rightarrow$ $O(\log n)$-APPROX. FOR MIS IN (α, β)-PLG FOR $\beta = 1$

QUESTION APPROXIMATION LOWER BOUND?

THE CASE $\beta = 1$

- START FROM ALON ET AL. '95

  LOWER BOUND Δ^{ε} FOR MIS

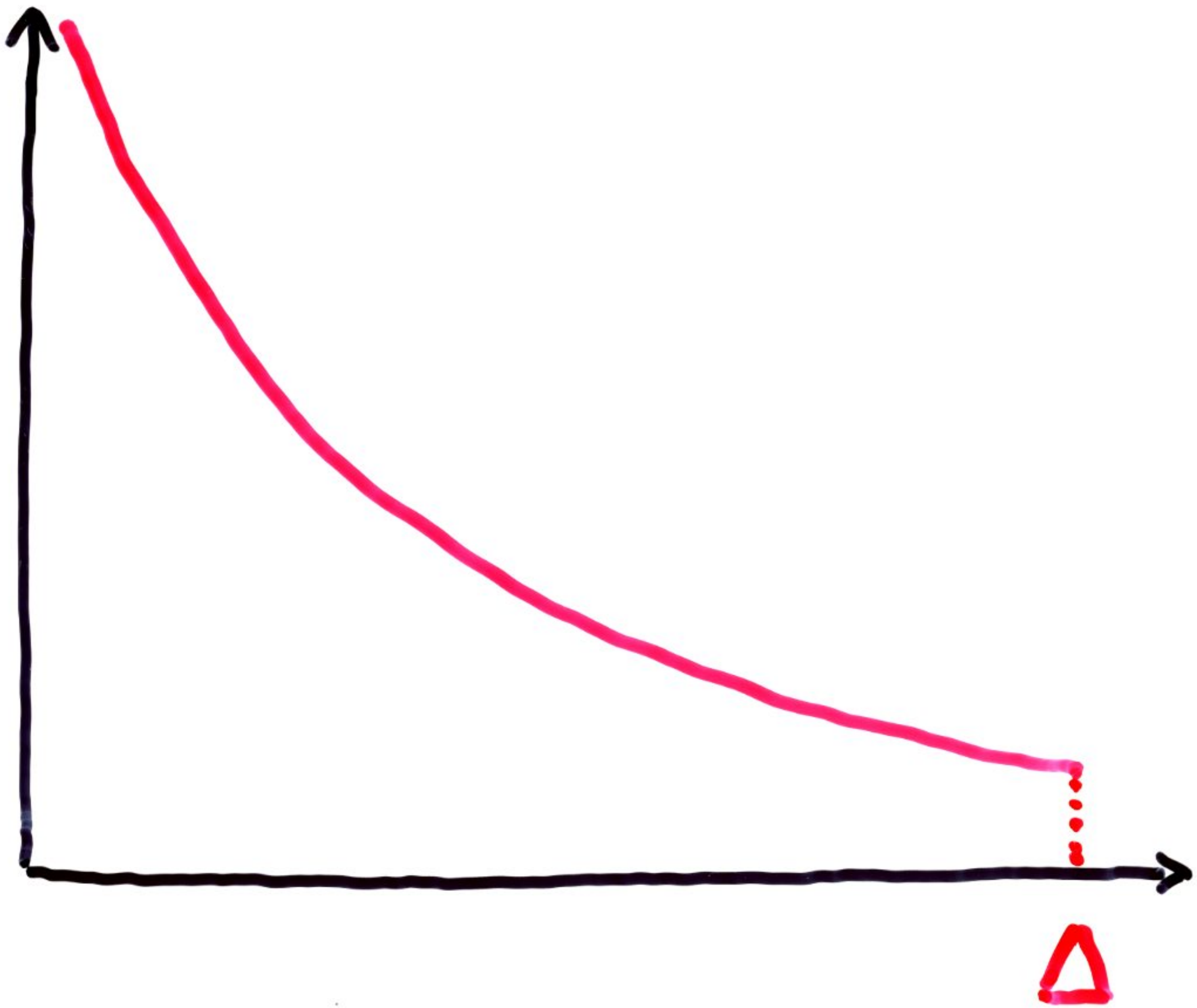
  IN GRAPHS WITH MAX. DEGREE $\leq \Delta$

THE CASE $\beta = 1$

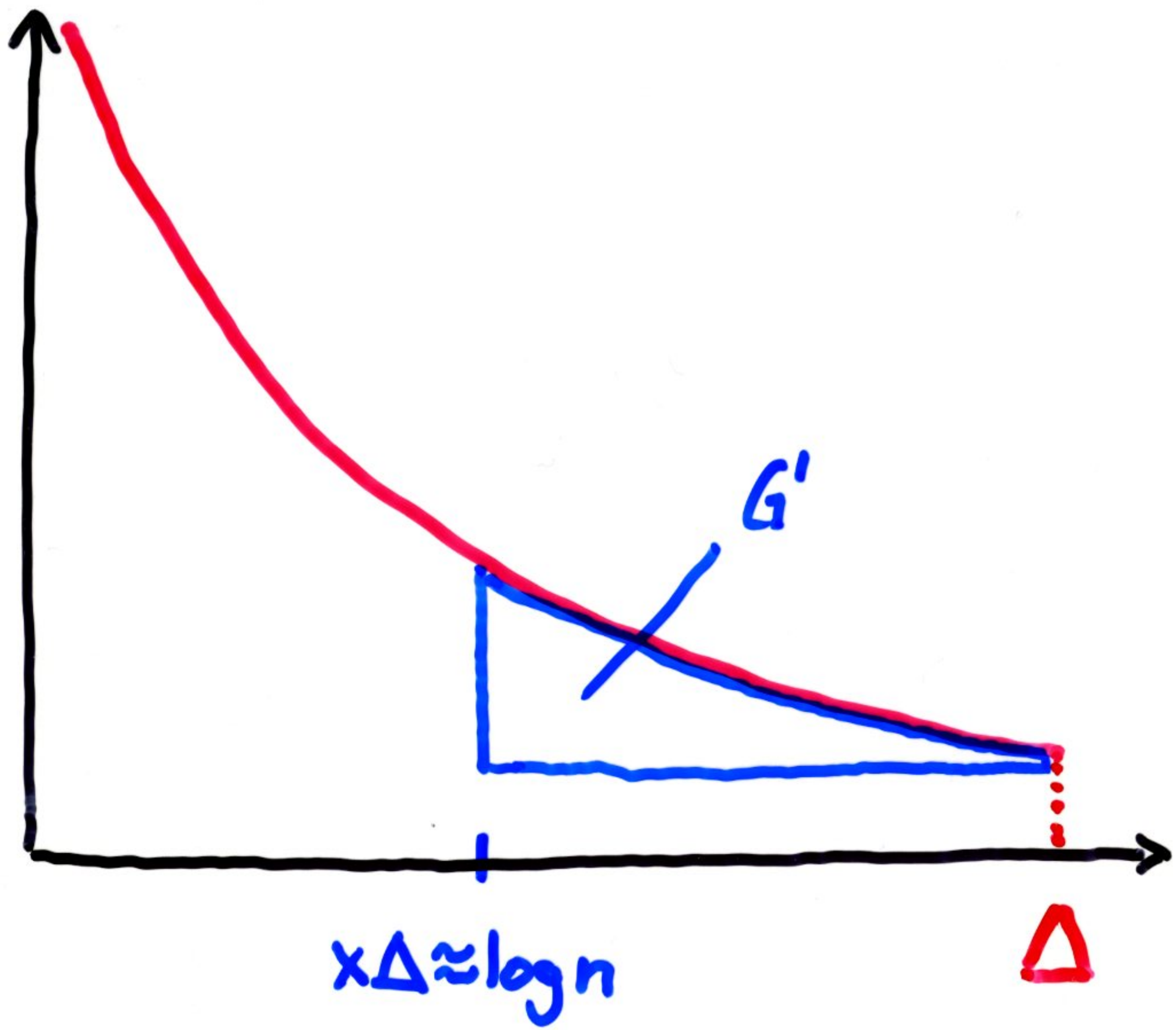
- START FROM ALON ET AL. '95
  LOWER BOUND Δ^{ε} FOR MIS
  IN GRAPHS WITH MAX. DEGREE $\leq \Delta$

- SHOW: THIS ALSO HOLDS FOR $\Delta = \log n$

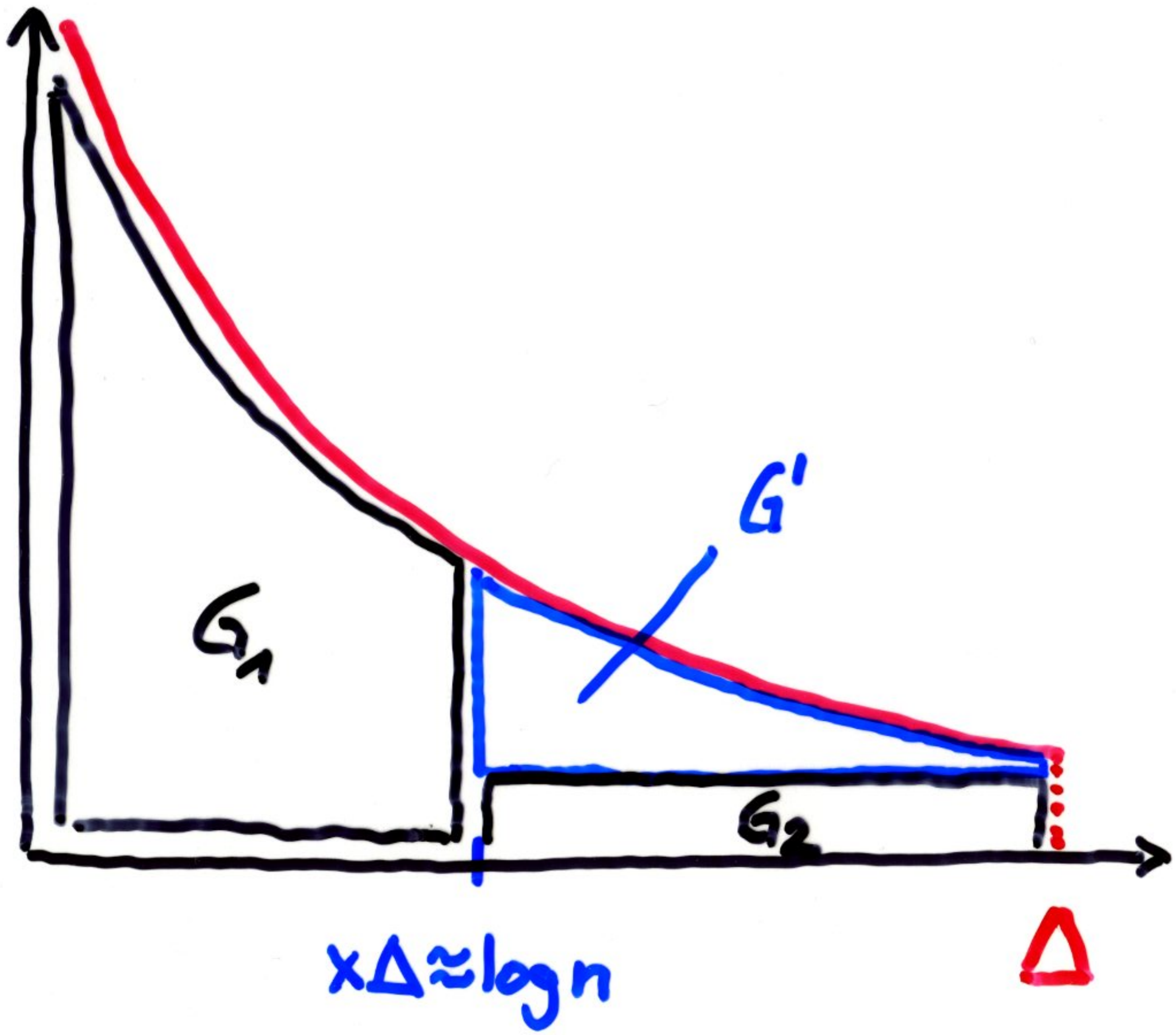
$\beta = 1$: THE EMBEDDING



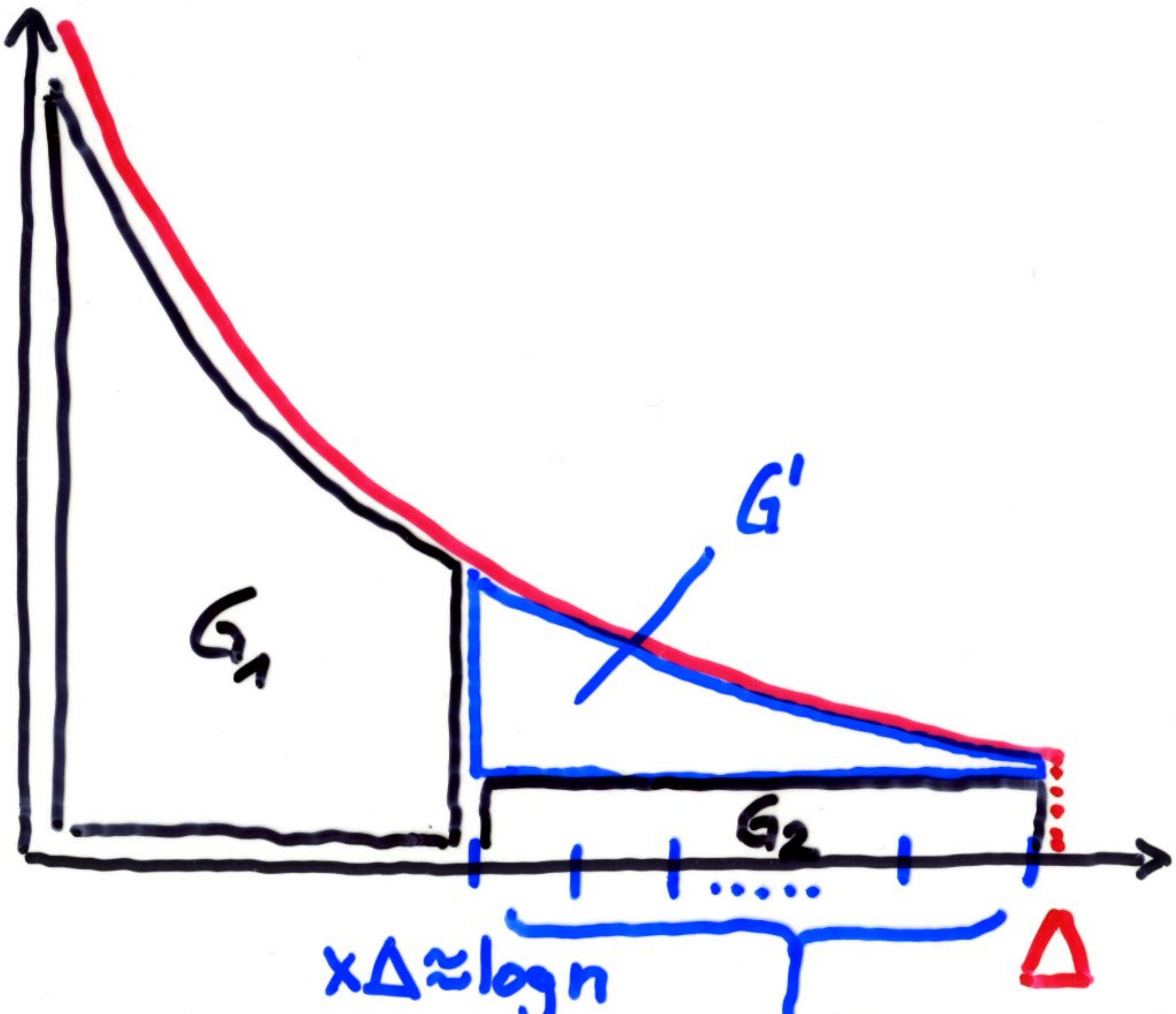
$\beta = 1$: THE EMBEDDING



$\beta = 1$: THE EMBEDDING



$\beta = 1$: THE EMBEDDING



▸ SPLIT INTO $\alpha = \log n$ PARTS

THE CASE $\beta=1$

$\Downarrow$

THEOREM

FOR $\beta=1$, THERE EXISTS $\varepsilon>0$ SUCH THAT MIS IN (α,β)-PLG IS NP-HARD TO APPROX. WITHIN $(\log n)^{\varepsilon}$.

OPEN PROBLEMS

- LOWER BOUND $(\log n)^{1-\varepsilon}$ FOR $\beta = 1$?

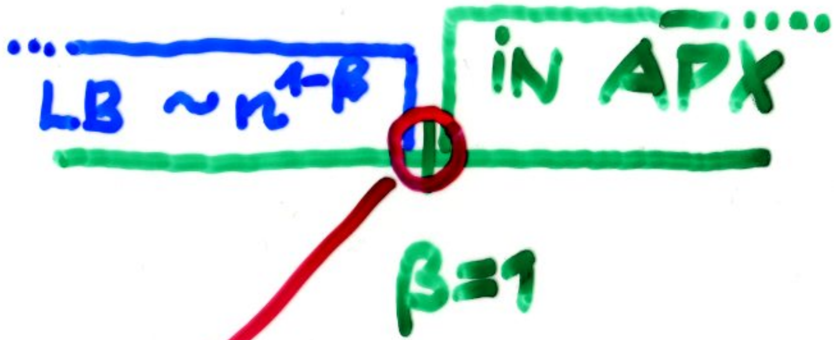
OPEN PROBLEMS

- LOWER BOUND $(\log n)^{1-\varepsilon}$ FOR $\beta = 1$?

- FUNCTIONAL CASES ?

OPEN PROBLEMS

- LOWER BOUND $(\log n)^{1-\varepsilon}$ FOR $\beta = 1$?
- FUNCTIONAL CASES ?



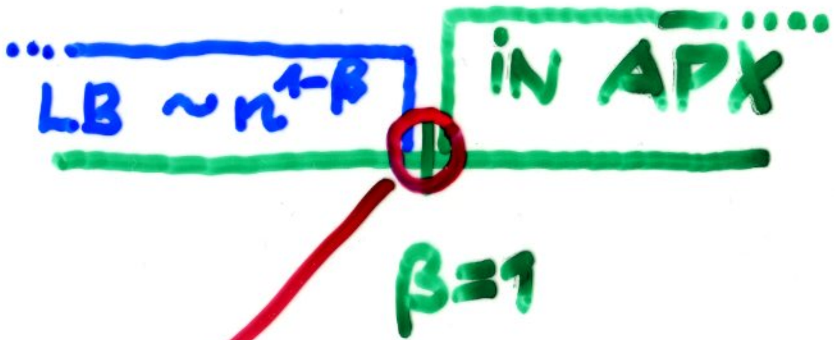
LOWER $(\log n)^{\varepsilon}$

UPPER $\log n$

OPEN PROBLEMS

- LOWER BOUND $(\log n)^{1-\varepsilon}$ FOR $\beta=1$?

- FUNCTIONAL CASES ?



LOWER $(\log n)^{\varepsilon}$

UPPER $\log n$

WHAT ABOUT

$$\beta = 1 - \frac{1}{f(n)}$$

$$\beta = 1 + \frac{1}{f(n)} \quad ?$$

REFERENCES

- M. HAUPTMANN, M. KARPINSKI
  ON THE APPROXIMABILITY OF INDEPENDENT SET PROBLEM ON POWER LAW GRAPHS
  COMPUTING RESEARCH REPOSITORY, arXiv: 1503.02880, 2015

MIS ON POWER LAW GRAPHS:

- A. FERRANTE, G. PANDURANGAN, K. PARK
  ON THE HARDNESS OF OPTIMIZATION IN POWER-LAW GRAPHS
  TCS 393, pp. 220-230, 2008

- Y. SHEN, D.T. NGUYEN, Y. XUAN, M.T. THAI
  NEW TECHNIQUES FOR APPROXIMATING OPTIMAL SUBSTRUCTURE PROBLEMS IN POWER LAW GRAPHS
  TCS 447, 107-119, 2012

POWER LAW GRAPHS

- R. ALBERT, H. JEONG, A.L. BARABASI
  THE DIAMETER OF THE WORLD WIDE WEB
  NATURE 401, pp. 130-131, 1999

- A.L. BARABASI
  EMERGENCE OF SCALING IN RANDOM NETWORKS
  SCIENCE 10, pp. 509-512, 1999

- W. AIELLO, F. CHUNG, L. LU
  A RANDOM GRAPH MODEL FOR MASSIVE GRAPHS
  PROC. 32ND ACM STOC, pp. 171-180, 2000

MAX CLIQUE / MAX INDEPENDENT SET

- N. ALON, U. FEIGE, A. WIGDERSON, D. ZUCKERMAN
  DERANDOMIZED GRAPH PRODUCTS
  COMPUTATIONAL COMPLEXITY 5(1), 60-75, 1995
- U. FEIGE, S. GOLDWASSER, L. LOVAŚZ, S. SAFRA, M. SZEGEDY
  INTERACTIVE PROOFS AND THE HARDNESS OF APPROXIMATING CLIQUES, JACM 43, 268-292, 1996

- J. HÅSTAD, CLIQUE IS HARD TO APPROXIMATE WITHIN $n^{1-\varepsilon}$, ACTA MATHEMATICA 182, 105-142, 1999
- S. KHOT, IMPROVED INAPPROXIMABILITY RESULTS FOR MAX CLIQUE, CHROMATIC NUMBER AND APPROXIMATING GRAPH COLORING
  PROC. 42ND IEEE FOCS, PP. 600-609, 2001
- D. ZUCKERMAN
  LINEAR DEGREE EXTRACTORS AND THE INAPPROXIMABILITY OF MAX CLIQUE AND CHROMATIC NUMBER
  THEORY OF COMPUTING 3, 103-128, 2007