Approximability of Combinatorial Optimization Problems on Power Law Networks

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b-it Bonn-Aachen International Center for Information Technology

First Observation

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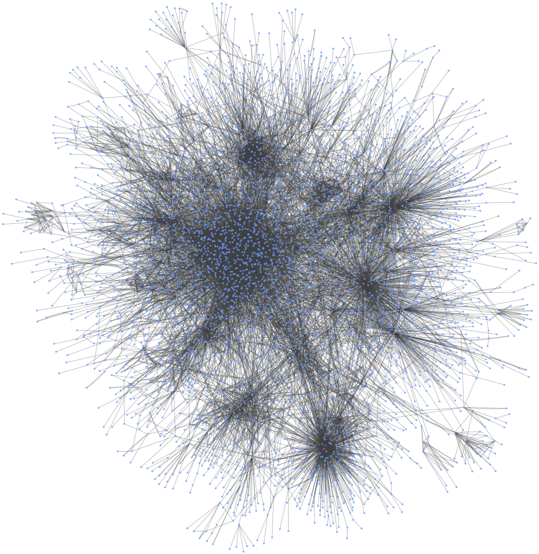
— Real world networks are not random, they have very small diameter and they possess a power law distribution of node degrees

Biological Networks

Example:
Protein interactions of *Arabidopsis Thaliana*



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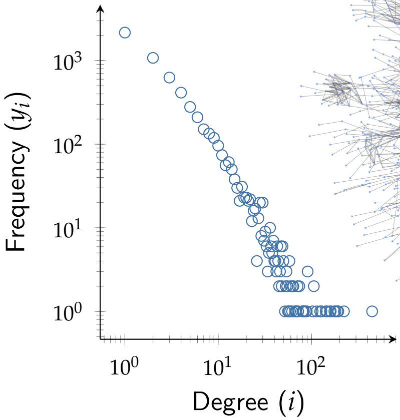
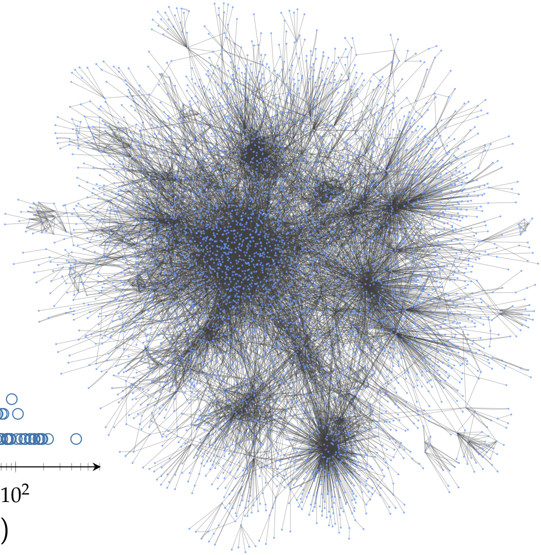
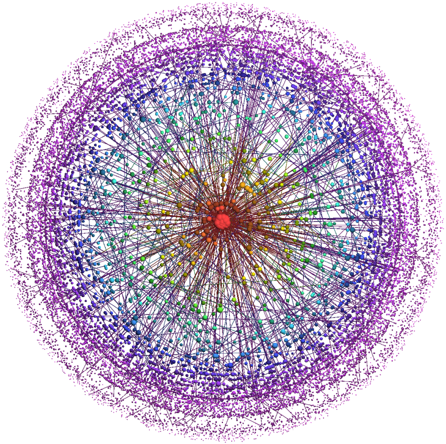




Technological Networks

Example:
Network of *Internet Routers*



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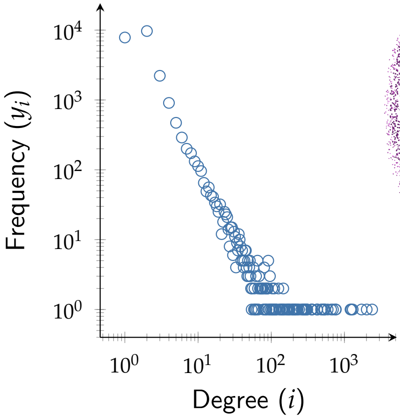
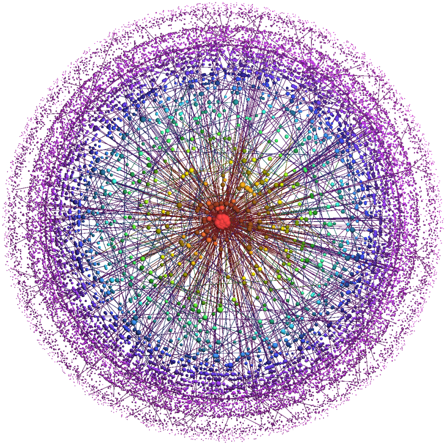
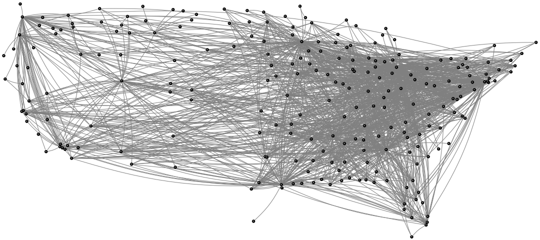




Example:
Airport Network in the United States



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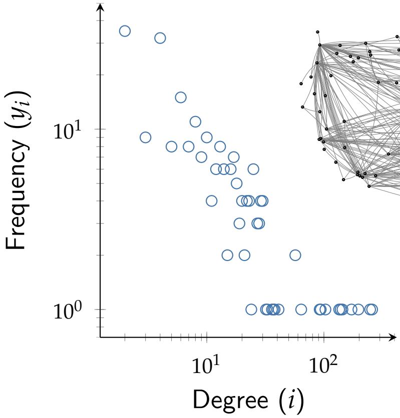
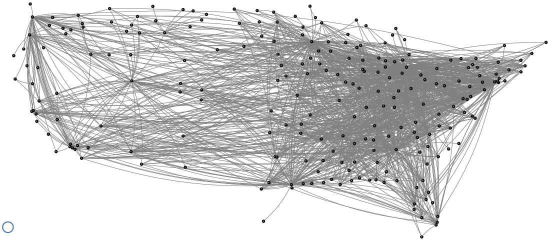
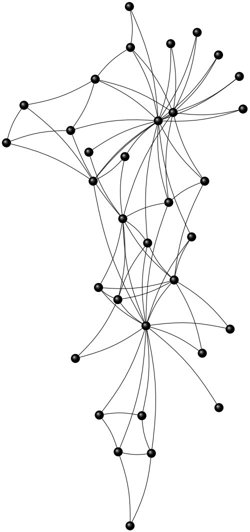




Example:
Contact Network of
karate club members



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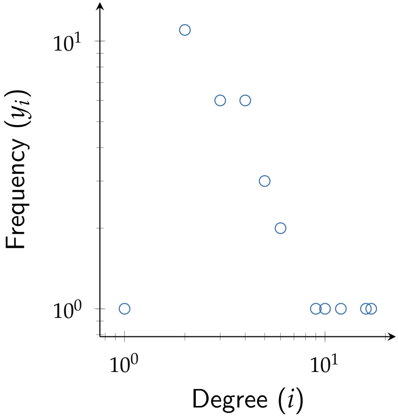

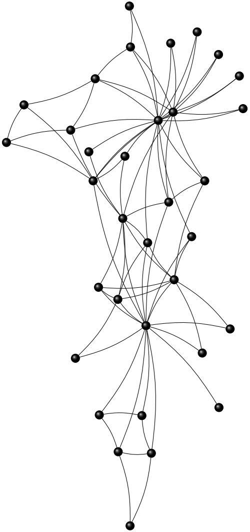

Power Law Degree Distribution

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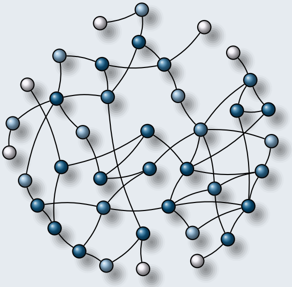
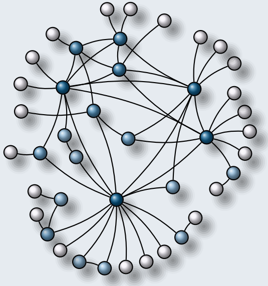
- Uniform random graph vs. power law random graph
- Number of nodes y_i having degree i: $y_i \sim c \cdot i^{-\beta}$



Erdős-Rényi Random Graph



Power Law Graph (PLG)

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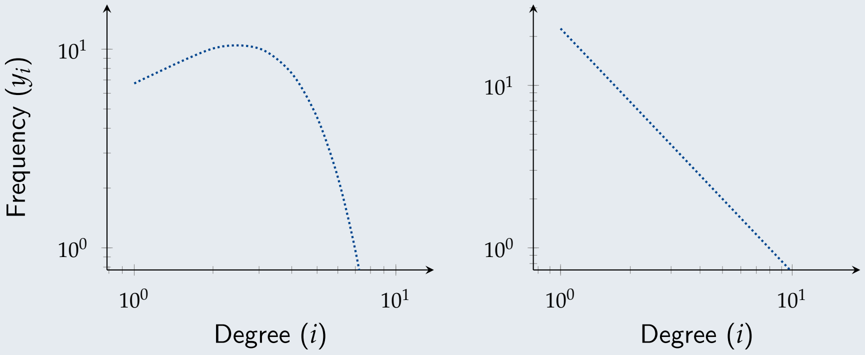

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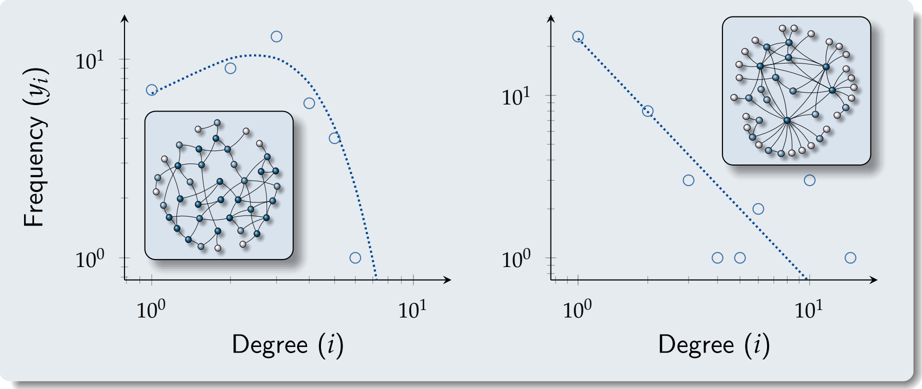

Motivation

— *The study of combinatorial optimization problems on real world networks is motivated by applications*

Example: Dominating Set Problem

Minimum dominating set problem in real world networks:

- Optimal sensor or server placement in wireless mobile networks
- Search for key players or nodes in social networks

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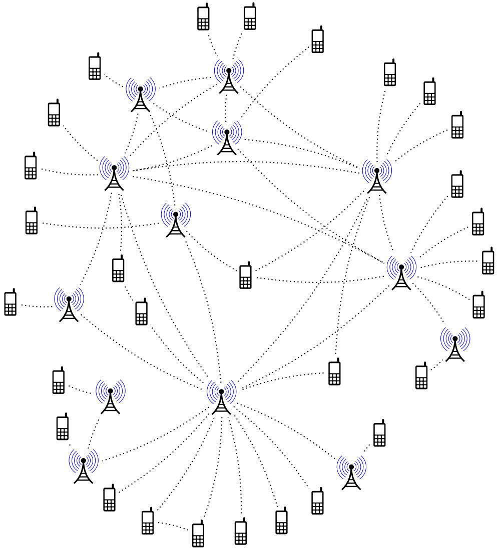

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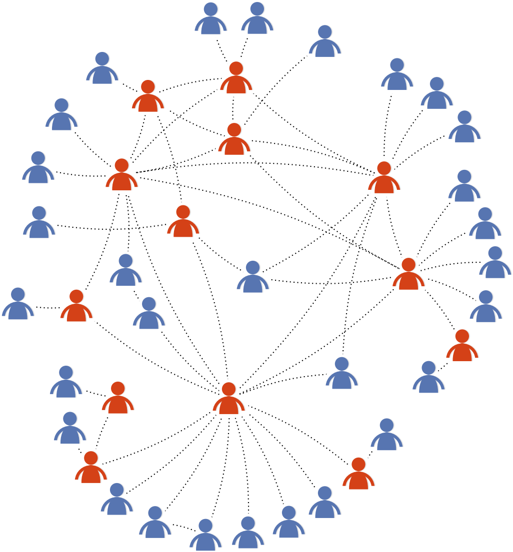

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Other Properties

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— Real world networks display a number of other unique and characteristic topological properties

Small Worlds

- Real world networks behave like “Small Worlds”
- Existence of bridging links across the network

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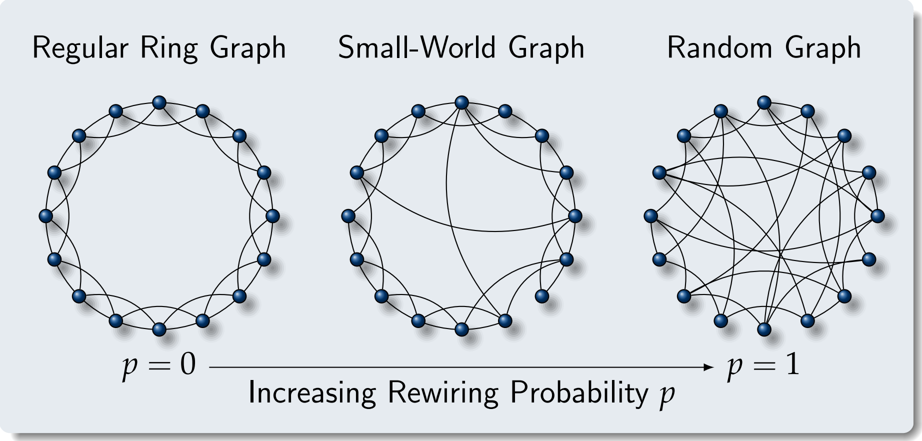

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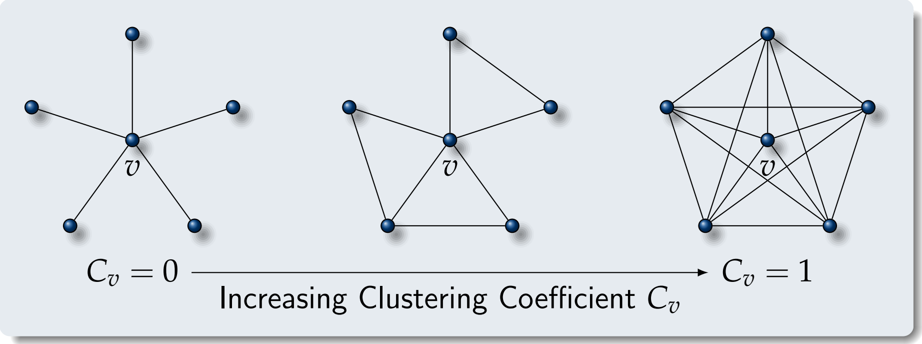

Hyperbolicity

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- Real world networks have embedded hyperbolic geometries
- Relates to Gromov's four-point condition for δ-hyperbolicity of a metric space

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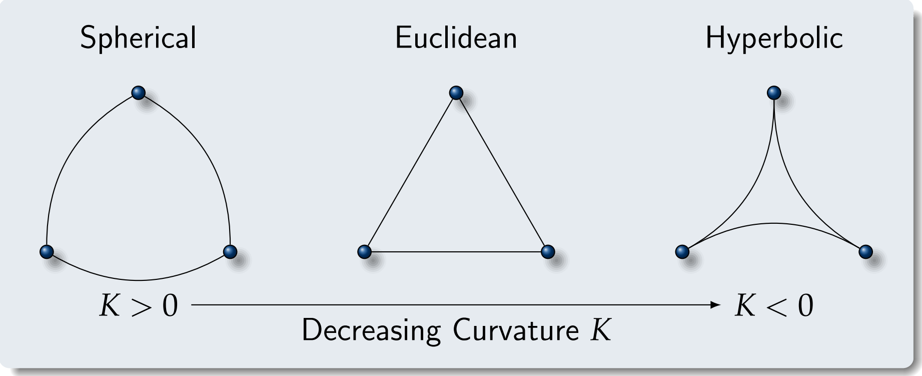

Modeling

> — *There exists a large number of generating models for power law graphs*

Evolving random model for PLG's:

- The Preferential Attachment Model *(Barabási and Albert, 1999)*

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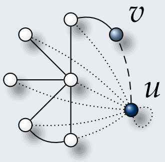

After adding u, probability that u connects to some vertex v:

$$\Pr(\{u,v\}) = \begin{cases} \deg(v)/\sum_i \deg(v_i) - 1 & u \neq v \\ 1/\sum_i \deg(v_i) - 1 & u = v \end{cases}$$

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Static random model for PLG's:

- The $\mathcal{G}_{\alpha,\beta}$ Model or ACL Model *(Aiello, Chung, and Lu, 2001)*

Ensures power-law degree distribution by fixing a degree sequence $(y_1, y_2, \ldots, y_\Delta)$ via two parameters α, β and then taking the space of random multigraphs with this degree sequence

Static Degree Sequences

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ACL Model for PLG's

Definition of the ACL Model $\mathcal{G}_{\alpha,\beta}$:

- For each $1 \leqslant i \leqslant \Delta = \lfloor e^{\alpha/\beta} \rfloor$,

$$y_i = \begin{cases} \left\lfloor \frac{e^\alpha}{i^\beta} \right\rfloor & \text{if } i > 1 \text{ or } \sum_{i=1}^{\Delta} \left\lfloor \frac{e^\alpha}{i^\beta} \right\rfloor \text{ is even} \\ \lfloor e^\alpha \rfloor + 1 & \text{otherwise} \end{cases}$$

- α is the logarithm of the network size, β is the log-log growth rate

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ACL Model for PLG's

- Number of vertices:

$$n = \sum_{i=1}^{\Delta} \left\lfloor \frac{e^{\alpha}}{i^{\beta}} \right\rfloor \approx \begin{cases} \zeta(\beta)\, e^{\alpha} & \text{if } \beta > 1 \\ \alpha\, e^{\alpha} & \text{if } \beta = 1 \\ \frac{e^{\alpha/\beta}}{1-\beta} & \text{if } 0 < \beta < 1 \end{cases}$$

- Number of edges:

$$m = \frac{1}{2} \sum_{i=1}^{\Delta} i \left\lfloor \frac{e^{\alpha}}{i^{\beta}} \right\rfloor \approx \begin{cases} \frac{1}{2}\zeta(\beta-1)\, e^{\alpha} & \text{if } \beta > 2 \\ \frac{1}{4}\alpha\, e^{\alpha} & \text{if } \beta = 2 \\ \frac{1}{2}\frac{e^{2\alpha/\beta}}{2-\beta} & \text{if } 0 < \beta < 2 \end{cases}$$

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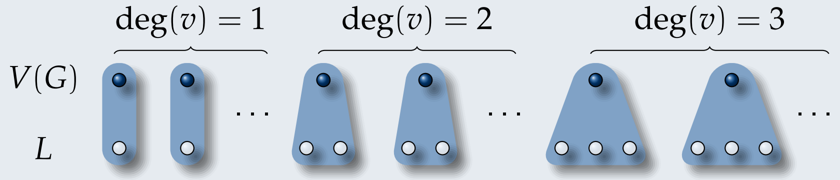
ACL Random Model

The distribution of graphs $G \in \mathcal{G}_{\alpha,\beta}$ over a sequence $(y_1, y_2, \ldots, y_\Delta)$ or $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ is generated as follows:

ACL Random Model

1. Generate set L of $\deg(v)$ distinct copies for each vertex $v \in V(G)$
2. $M :=$ *random matching* on the elements of L
3. For $u, v \in V(G)$ number of edges $\{u, v\}$ equals number of edges $m \in M$ that join copies of u and v



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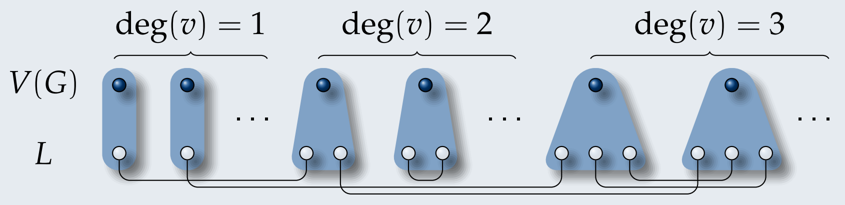

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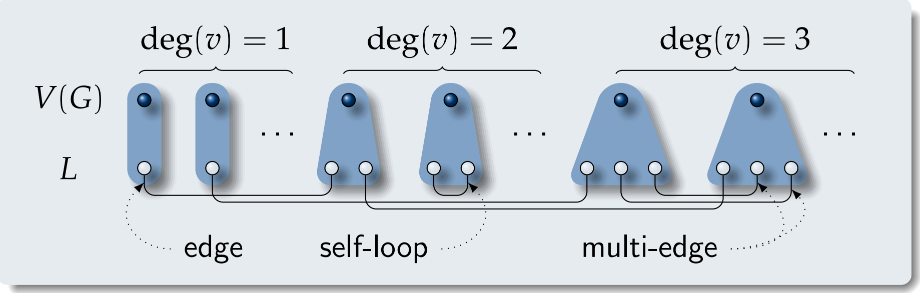

Overview of Main Results

Presented here:

- Approximation lower bounds for MINIMUM DOMINATING SET (MIN-DS) in connected PLG's
- Approximation upper bounds for MINIMUM VERTEX COVER (MIN-VC) in random PLG's

Techniques:

- Connected Embedding Approximation-Preserving (CEAP) reductions
- Transforming hardness results for bounded occurrence CSP's and SET COVER

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Techniques:

- LP-relaxation and deterministic rounding algorithm
- Upper and lower bounds on the size of half-integral solutions in random PLG's

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- Approximation upper bounds for MIN-DS for $\beta > 2$

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Lower bound technique:

CEAP reductions (high level view)

- Embed bounded occurrence CSP and SET COVER reduction instances G' into PLG $G_{\alpha,\beta} \in \mathcal{G}_{\alpha,\beta}$
- Achieve connectivity with reasonable cut sizes between G' and $G_{\alpha,\beta} \setminus G'$
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Bounded Occurrence CSP Paradigm

Method: Bounded degree amplifier graphs

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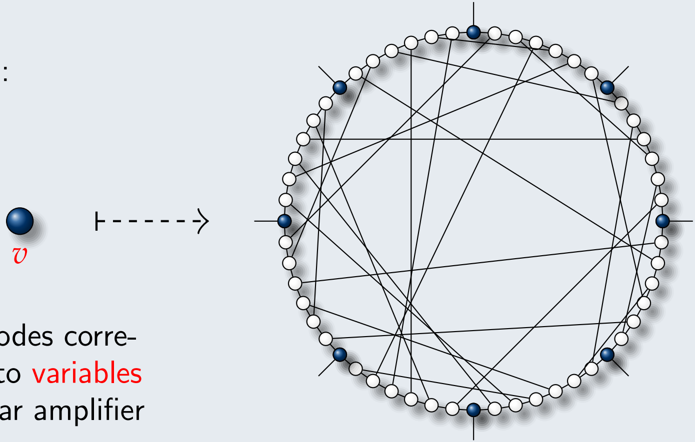
Basic Idea:



Replace nodes corresponding to variables by 3-regular amplifier

From bounded occurrence CSP's to vertex covers:

- Reduce bounded occurrence HYBRID (equations with 2 and 3 variables) to MIN-VC on degree d bounded graphs (d-MIN-VC)
  - Yields explicit lower bounds of $\frac{103}{102}$ for $d = 3$ and $\frac{55}{54}$ for $d = 4, 5$ *(Berman and Karpinski, 2003)*
  - For larger d assuming UGC: $2 - (2 + o(1))\frac{\log \log d}{\log d}$ *(Austrin, Khot, and M. Safra, 2009)*
- d-MIN-VC serves as starting point for our CEAP reduction to MIN-VC on PLG's

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Set Cover Paradigm

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From set covering to dominating sets:

- $G_{U,\mathcal{S}}$ instances will serve as starting point for our CEAP reduction to MIN-DS on PLG's

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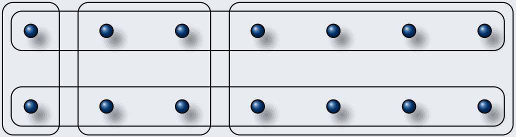
From set covering to dominating sets:

A Set Cover instance $(U, \mathcal{S})$



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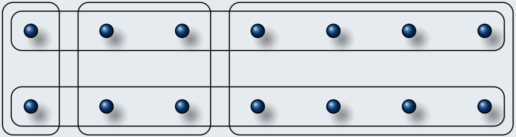

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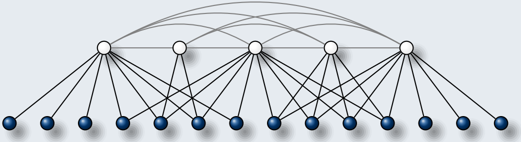

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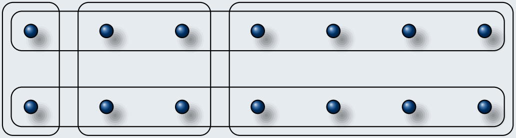

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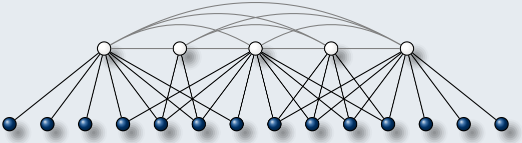

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Approximation Lower Bounds for MINIMUM DOMINATING SET on Connected Power Law Graphs

Minimum Dominating Set

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Definition (Min-DS)

Input: A graph $G = (V, E)$

Output: A subset $D \subseteq V$ such that for each vertex $v \in V$ either $v \in D$ or $D \cup N(v) \neq \emptyset$

Objective: Minimize $|D|$

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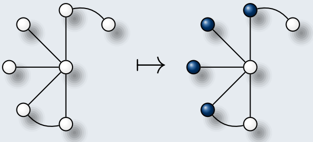

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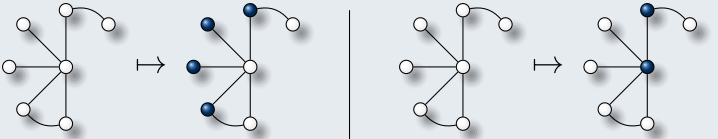

Dominating Set | Minimum Dominating Set

Minimum Dominating Set

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Approximability on general graphs:

- Upper bound: $\ln n$ *(Johnson, 1974; Lovász, 1975)*
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Minimum Dominating Set

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Definition (Min-DS)

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Simple PLG's	General PLG's
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Hardness of Min-DS on PLG's

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Open Questions

- Is MINIMUM DOMINATING SET $\mathcal{NP}$-hard and $\mathcal{APX}$-hard on connected PLG's?
- Can we close the gap between the constant lower bounds on PLG's and the general logarithmic lower bound?
- Can we extend the results to the range $\beta \in [0, 1]$?

Theorem *(Gast, Hauptmann, and Karpinski, 2012)*

For all $\beta \in [0+\varepsilon, 2]$ and $\varepsilon > 0$, MIN-DS is hard to approximate within $\Omega(\ln(n) - c_\beta)$ on connected PLG's

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Embedding technique (CEAP reduction):

- Map $G_{U,\mathcal{S}}$ to $G_{\alpha,\beta}$ via scaling construction connecting to a multigraph wheel W
  - Number of edges between $G_{U,\mathcal{S}}$ and W is $O(\min\{|G_{U,\mathcal{S}}|, |W|\})$
- Vertex set Γ separates $G_{U,\mathcal{S}}$ from $G_{\alpha,\beta} \setminus G_{U,\mathcal{S}}$
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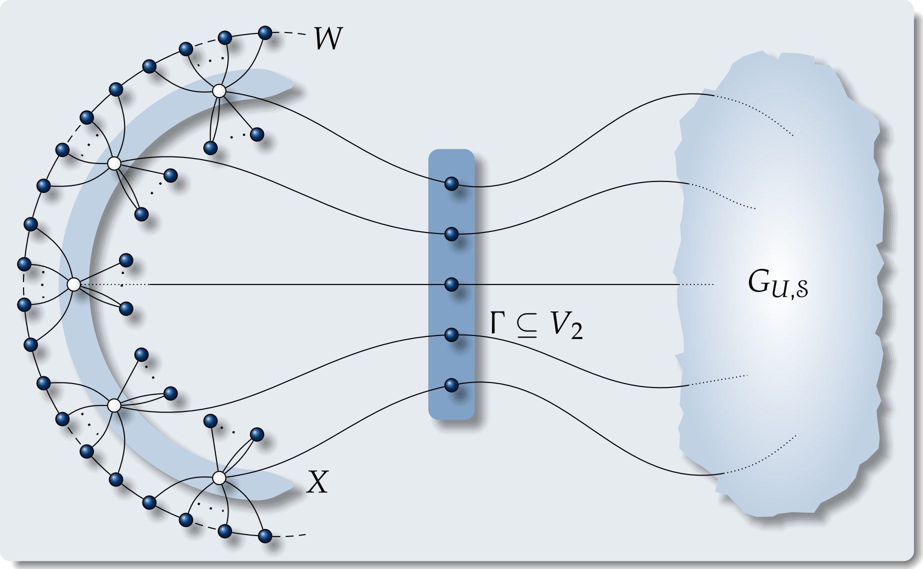

Phase Transitions

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Observation

For $\beta > 2$, Min-DS on $\mathcal{G}_{\alpha,\beta}$ PLG's is in $\mathcal{APX}$

Analysis of the phase transition:

- Study of functional case $\beta_f = 2 + \frac{1}{f(n)}$
  - Hard to approximate within $\Omega(\ln(n) - c_\beta)$ for $f(n) = \omega(\log(n))$
  - In $\mathcal{APX}$ for $f(n) = o(\log(n))$ (!)

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Approximation Upper Bounds for
MINIMUM VERTEX COVER on
Random Power Law Graphs

Minimum Vertex Cover Problem

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Input: A graph $G = (V, E)$

Output: A subset $C \subseteq V$ such that each edge $\{u, v\} \in E$ has at least one endpoint in C

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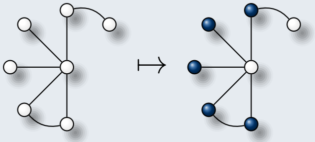

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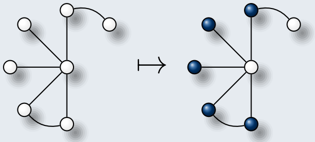

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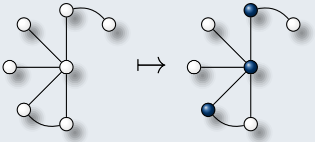

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There exists practical evidence that MIN-VC is easier to approximate on PLG's

- The greedy algorithm often outperforms the 2-approximation algorithm *(Park and Lee, 2001)*
- Achieves average ratios of ~ 1.02 on real world network topologies *(M. O. Da Silva, Gimenez-Lugo, and M. V. G. Da Silva, 2013)*

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Approximation of Min-VC on PLG's

Open Question

Can we give provable guarantees that MIN-VC is easier to approximate on PLG's?

Theorem *(Gast and Hauptmann, 2012)*

There exists an approximation algorithm for MIN-VC on random $\mathcal{G}_{\alpha,\beta}$ PLG's with expected approximation ratio

$$\rho \leqslant 2 - \frac{\zeta(\beta) - 1 - \frac{1}{2^\beta}}{2^\beta \zeta(\beta - 1)\zeta(\beta)}$$

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Consider the following LP-Relaxation for Min-VC:

Half Integral Solutions

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\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} w_i x_i, \\
\text{subject to} \quad & x_i + x_j \geqslant 1, \quad \text{for all edges } e = \{v_i, v_j\}, \\
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\end{aligned}
$$

- There always exists optimal solution which is half-integral, i.e. $\forall i : x_i \in \{0, 1/2, 1\}$ and $v_i \in V_0, V_{1/2}, V_1$, respectively
- A half-integral solution can be computed in polynomial time (using algorithm for Min-VC or Perfect Matching in bipartite graphs)

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Start with half-integral solution $x : V \to \{0, 1/2, 1\}$

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Overall approximation ratio as convex combination of ratio $3/2$ on V' and ratio 2 on $V \setminus V'$

Open Problems and Further Research

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- Still improving on the presented results
  - Investigating the gap between upper and lower approximation bound for MIN-VC on PLG's
  - Improving upper bounds for MIN-DS on PLG's when $\beta \leqslant 2$ (in random or quasi-random models)
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