METRIC CONSTRUCTION FOR PATH COUPLING, RAPID MIXING, AND APPROXIMATE COUNTING.

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(JOINT WORK WITH M. BORDEWICH, M. DYER)

APPROXIMATE COUNTING

OF "EXCEEDINGLY" HARD PROBLEMS IN STATISTICAL PHYSICS, COMBINATORICS, GEOMETRY, NETWORK DESIGN, ...

(#P-HARD PROBLEMS)

A GENERIC TOKEN PROBLEM:

COUNTING INDEPENDENT SETS IN GRAPHS, AND HYPERGRAPHS, ARISE IN THE HARD-CORE MODEL OF A GAS WHERE A POSSIBLE PARTICLE SITE & ADJACENT SITES CANNOT BE SIMULT. OCCUP.

THE PROBLEM IS EQUIVALENT TO THE PROBLEM OF COUNTING SATISFYING ASSIGNMENTS IN MONOTONE EkSAT-FORMULAS

$$f = \bigwedge_{i \in A} C_i \quad \text{FOR}$$

$$C_i = \bigvee_{j=1}^{k} x_{i_j} .$$

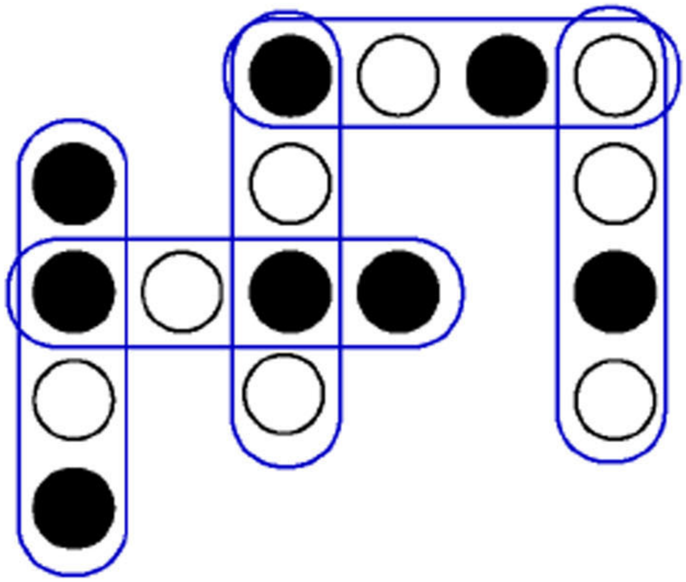
GIVEN A k-UNIFORM HYPERGRAPH

$H = (V, E)$,

$$E = \{h \mid h \subseteq V, |h| = k\}.$$

$I \subseteq V$ IS INDEPENDENT IF $h \nsubseteq I$ FOR ALL $h \in E$.

Ω IS THE SET OF ALL INDEPENDENT SETS OF H. THE PROBLEM: APPROXIMATE $|\Omega|$.



INDEPENDENT SET

I (BLACK VERTICES)
IN A 4-UNIFORM
HYPERGRAPH.

APPROXIMATING A COUNTING PROBLEM $f:\Sigma^* \to \mathbb{N}$ BY AN APPROX. SCHEME $\mathcal{A}$, GIVEN $\varepsilon > 0$, AN ERROR TOLERANCE,

$$\Pr\left[(1-\varepsilon)f(x) \le \mathcal{A}(x) \le (1+\varepsilon)f(x)\right] \ge \frac{3}{4}$$

$$\left(\ge 1-\delta,\ \delta = 10^{-5}\right)$$

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$(\geq 1-\delta,$
$\delta = 10^{-5})$

$\mathcal{A}$ IS FPRAS IF $\mathcal{A}$ RUNS IN $(|x| \varepsilon^{-1})^{O(1)}$ TIME.

BAD NEWS FOR INDEPENDENT SETS:

THERE IS NO FPRAS FOR COUNTING ISs EVEN FOR $K=2$.(*)

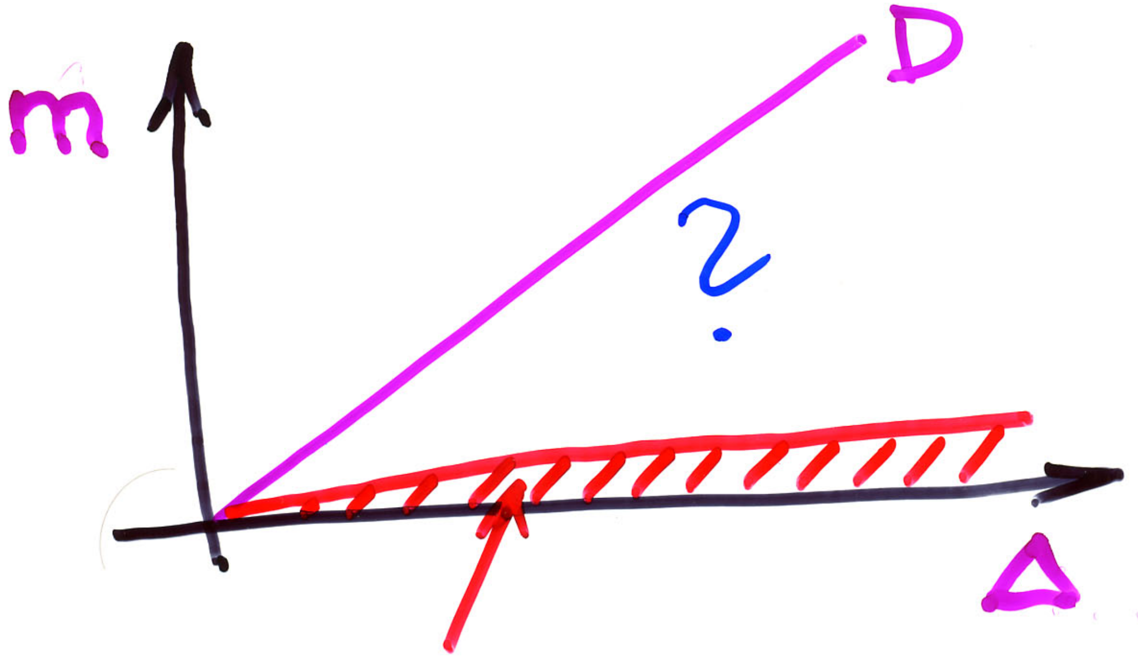
(PROOF BY REDUCTION FROM MAXIMUM INDEPEN. SET PROBLEM)

(*) UNLESS RP = NP.

DEPENDENCE ON DEGREE & DIMENSION?

THERE IS NO FPRAS FOR HYPERGRAPHS WITH MAXIMUM DEGREE Δ AND MINIMUM HYPEREDGE SIZE $m < 2\log(1+\Delta/694) - 1 = \Omega(\log \Delta)$.

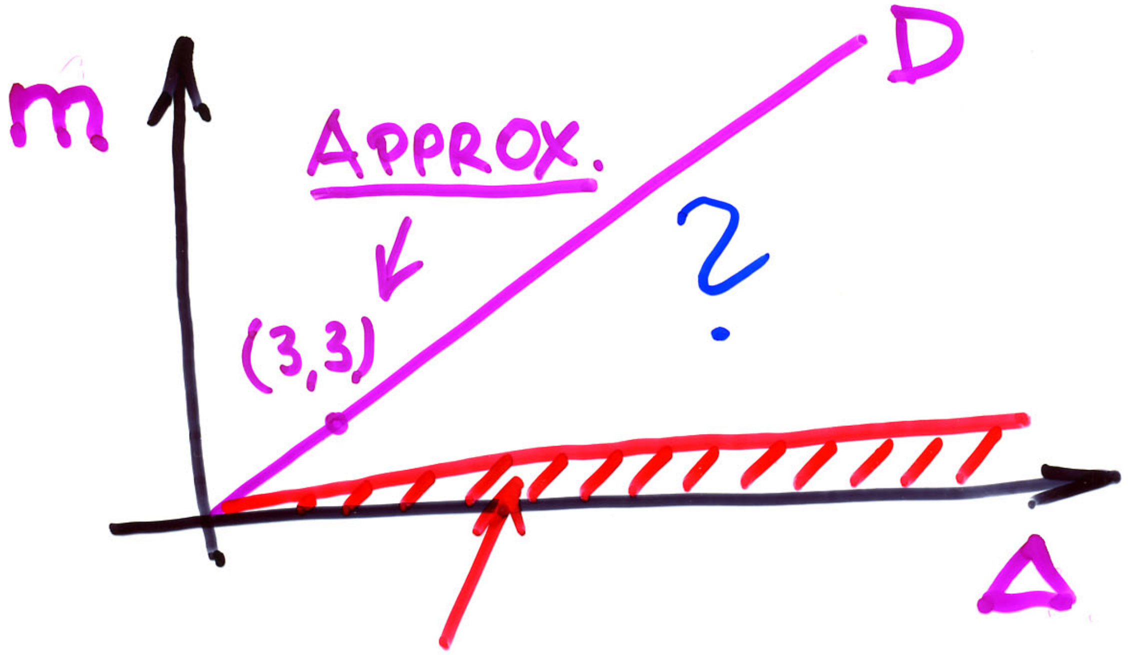
(BORDEWICH, DYER, K. '05)



APPROX. HARDNESS AREA.

PARTIAL ANSWER TO ?-AREA:

THERE IS AN FPRAS FOR $m \geq \Delta + 2 \geq 5$ OR $\Delta = 3$ AND $m \geq 2$.



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METHOD OF SOLUTION:

- SAMPLING FROM Ω (HARD-SET) ALMOST UNIFORMLY BY A MARKOV CHAIN $\mathcal{M}$ (NEED RAPID MIXING OF $\mathcal{M}$).
- USE ALMOST UNIFORM SAMPLING TO DESIGN AN FPRAS (SELF-REDUCIB.)

<u>METHOD</u> OF SOLUTION:

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CONSTRUCT $\mathcal{M} = (I_t)$ WITH STATE SPACE Ω AS FOLLOWS:

(i) CHOOSE $v \in_R V$;

(ii) WITH PROB. $\frac{1}{2}$ SET $\underbrace{I_t' = I_t - \{v\}}$,

WITH PROB. $\frac{1}{2}$ SET $\underbrace{I_t' = I_t \cup \{v\}}$;

(iii) IF $I_t' \in \Omega$ SET $\underbrace{I_{t+1} = I_t'}$

OTHERWISE $I_{t+1} = I_t$.

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$\mathcal{M}$ IS <u>EASILY</u> SEEN TO BE ERGODIC WITH UNIFORM STATIONARY DISTRIBUTION,

<u>LET</u> $p_t(I)$ BE THE <u>DISTRIBUTION</u> OF I_t, <u>THEN</u> FOR ALL $I \in \Omega$:

$$\boxed{\lim_{t\to\infty} p_t(I) = \frac{1}{|\Omega|}}.$$

<u>HOW</u> **FAST** TO <u>GET</u> <u>CLOSE</u> ENOUGH TO UNIFORM?

MEASURING RATE OF CONVERGENCE TO STATIONARITY:

TOTAL VARIATION DISTANCE d_{TV} BETWEEN P_t AND THE UNIFORM

$$d_{TV}(P_t, \pi) = \frac{1}{2} \sum_{I \in \Omega} \left| P_t(I) - \pi(I) \right|$$

<u>DEFINE</u> A COUPLING (I_t, J_t) FOR $\mathcal{M}$, I.E. A MARKOV CHAIN ON $\Omega \times \Omega$ SUCH THAT (I_t) AND (J_t) <u>ARE</u> MARGINALLY <u>FAITHFUL</u> <u>COPIES</u> OF $\mathcal{M}$.

COUPLING LEMMA (DOEBLIN, 1933).

$$d_{TV}(p_t, \pi) \leq \underline{P_R[I_t \neq J_t]}$$

FOR (J_t) <u>STARTING</u> WITH J_0 IN <u>STATIONARY D.</u>

FOR $\varepsilon > 0$, DEFINE THE MIXING TIME $\tau(\varepsilon)$ OF $\mathcal{M}$,

$$\tau(\varepsilon) = \min_t \{ d_{TV}(P_t, \pi) \leq \varepsilon \}$$

($\mathcal{M}$ IS RAPIDLY MIXING IF $\tau(\varepsilon)$ IS POLY. BOUNDED IN n AND $\log \varepsilon^{-1}$)

<u>ESTIMATES</u> FOR d_{TV} BY <u>USING</u> A METRIC $d(I_t, J_t)$ <u>DEFINED</u> ON $\Omega \times \Omega$, <u>TRYING</u> TO REDUCE IT IN <u>EXPECTATION</u> AT <u>EACH</u> STEP.

WHAT CAN WE PROVE FOR $\tau(\varepsilon)$ OF $\mathcal{M}$ ASSOCIATED WITH A HYPERGRAPH $H=(V,E)$ OF MAX. DEGREE Δ AND MIN. EDGE SIZE m, SAY BY THE DIRECT "IDENTITY" (I_t, J_t) COUPLING FOR $\mathcal{M}$?

TAKE $d = H$ FOR
H A **HAMMING** **METRIC**
ON Ω,

$$\underbrace{H(I, J)} = |I \setminus J| + |J \setminus I|$$

$=$ # OF VERTICES
AT **WHICH** I AND J
DIFFER.

THE **CHANGE** OF
EXPECTED HAMMING
DISTANCE IN ONE STEP
COULD BE $\frac{\Delta}{2n} - \frac{1}{n}$ (!)

PATH COUPLING METHOD (BUBLEY, DYER; 1997).

LET d BE AN INTEGER VALUED METRIC ON Ω. LET $S \subseteq \Omega \times \Omega$ SUCH THAT FOR ALL $(I_t, J_t) \in \Omega \times \Omega$ THERE EXISTS A PATH $I_t = Z_0, Z_1, \ldots, Z_r = J_t$ WITH $(Z_{i-1}, Z_i) \in S$ FOR $i = 1, \ldots, r$ AND

$$d(I_t, J_t) = \sum_{i=1}^{r} d(Z_{i-1}, Z_i)$$

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WITH VALUES IN $\{0, \ldots, D\}$

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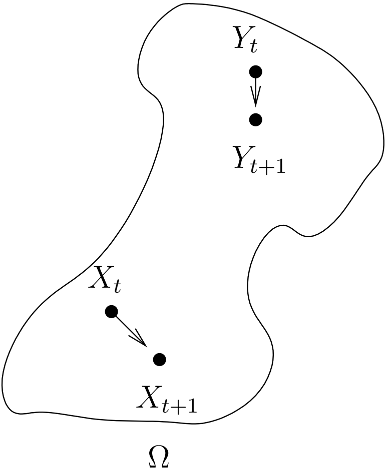
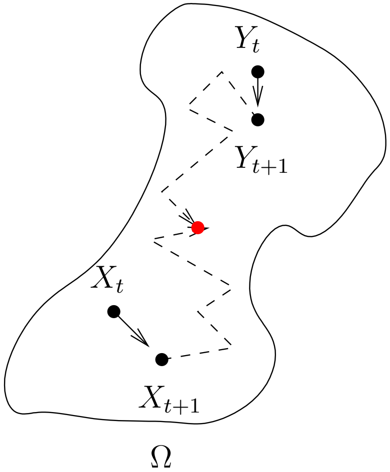

Y_t
Y_{t+1}
X_t
X_{t+1}
Ω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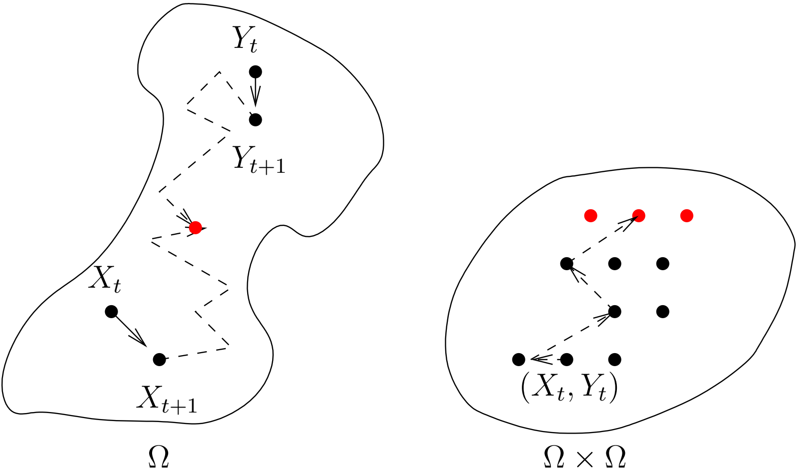
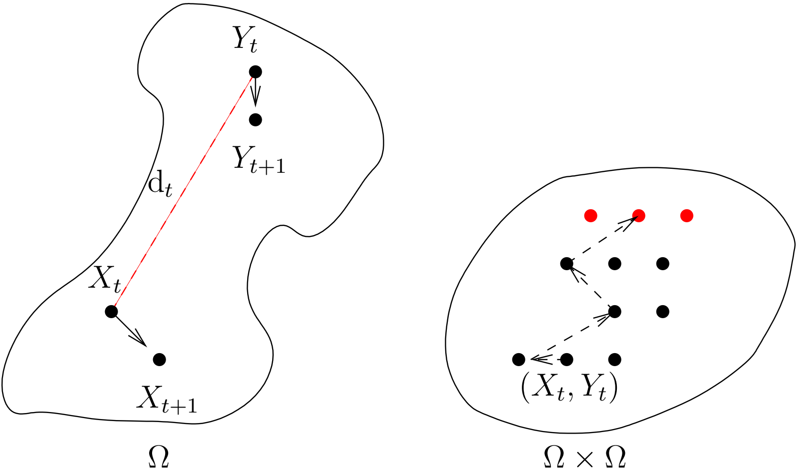

Y_t
Y_{t+1}
X_t
X_{t+1}
Ω
(X_t, Y_t)
$\Omega \times \Omega$


Y_t
Y_{t+1}
d_t
X_t
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Ω
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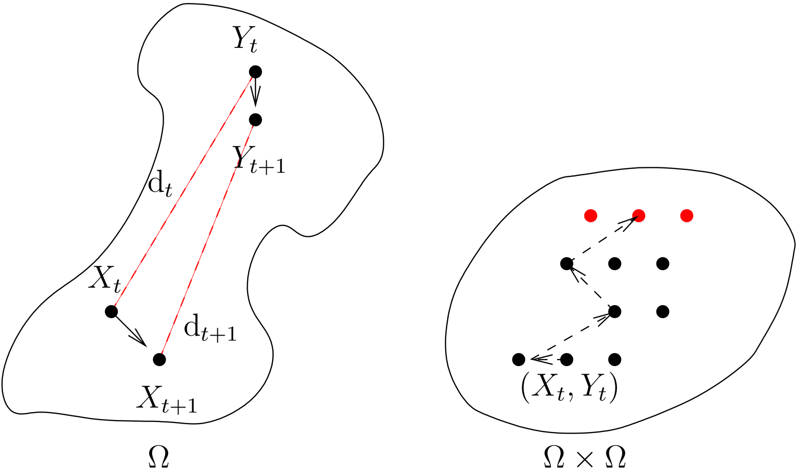
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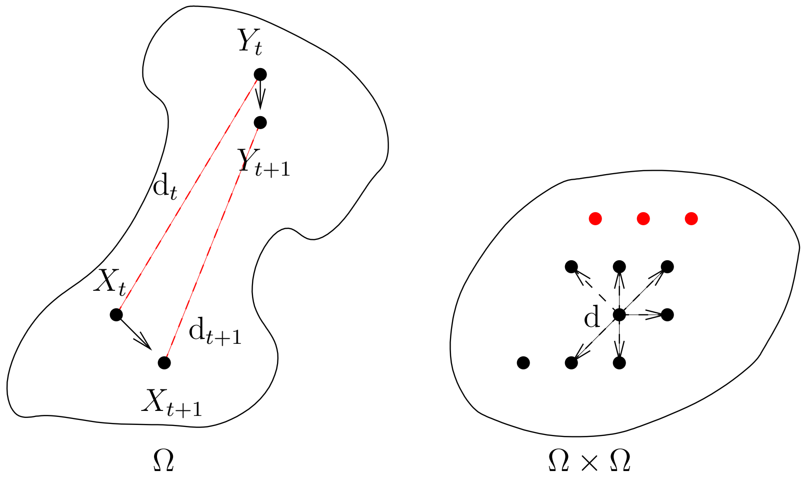
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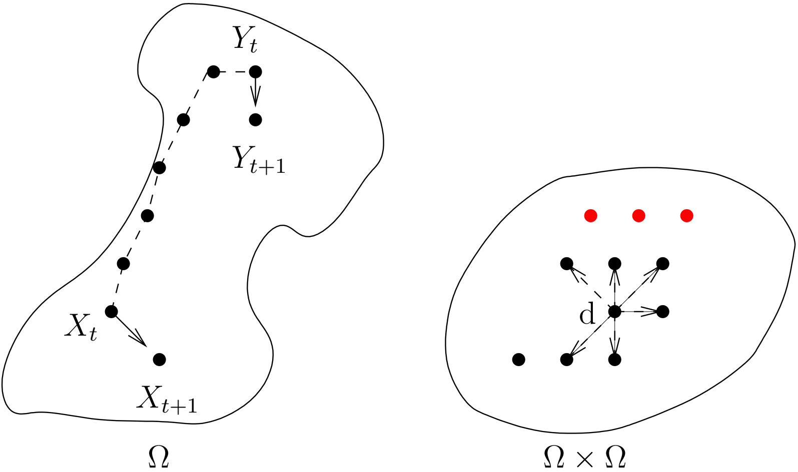

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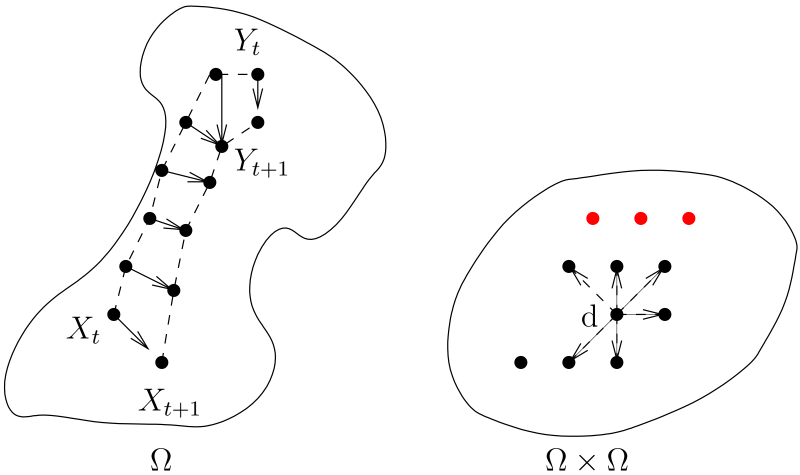
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DEFINE COUPLING ONLY FOR PAIRS IN S, AND THEN EXTEND IT TO A COUPLING FOR ANY PAIR VIA A SHORTEST PATH IN S.

THEOREM (BUBLEY, DYER).

IF $\mathbb{E}[d(I_{t+1}, J_{t+1})] \leq \beta d(I_t, J_t)$ FOR SOME $\beta < 1$ AND ALL $(I_t, J_t) \in S$,

THEN $$\tau(\varepsilon) \leq \frac{\ln(D\varepsilon^{-1})}{1-\beta}.$$

TAKE THE HAMMING METRIC H, AND DEFINE

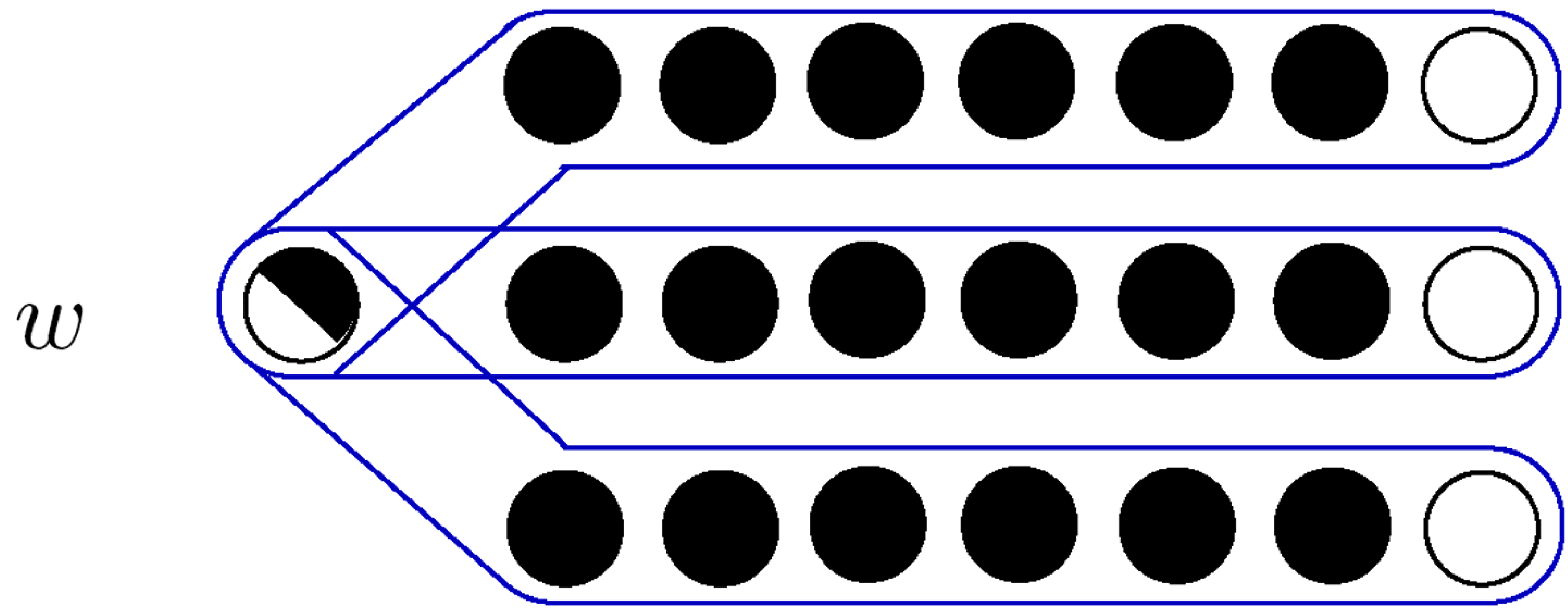
$$S = \{(I,J) \mid (I,J) \in \Omega \times \Omega, \; H(I,J) = 1\}.$$

$$\beta = \frac{2n + \Delta - 2}{2n}, \text{ AND}$$

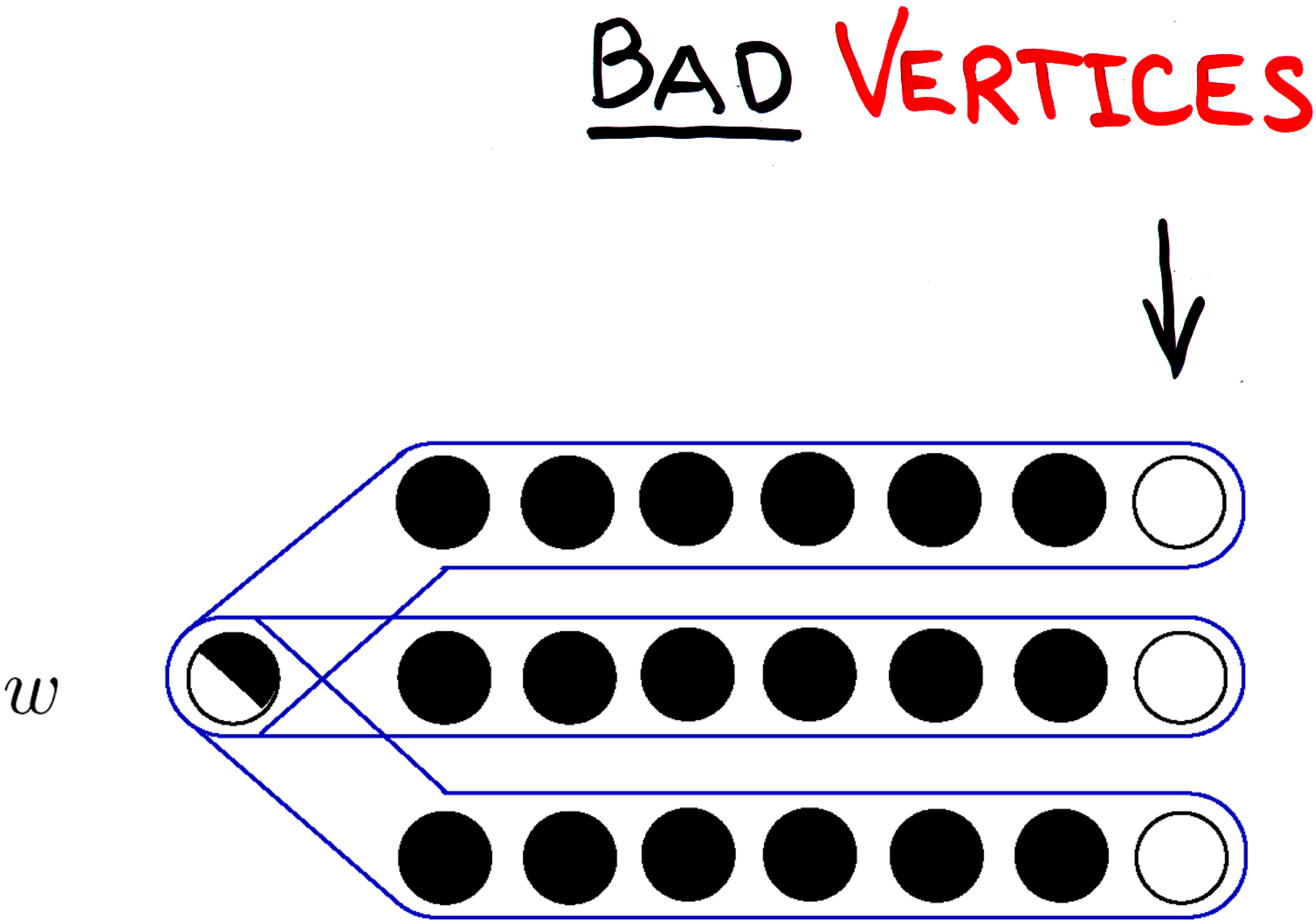
BY BUBLEY-DYER TH.

$$\tau(\varepsilon) \leq \frac{2n \ln(n\varepsilon^{-1})}{2 - \Delta},$$

NOT MUCH OF A USE, NEW IDEAS NEEDED.



BAD CONFIGURATION FOR $w \in I_t$ AND $w \notin J_t$.
△ VERTICES CAN BE INSERTED INTO J_t BUT NOT I_t.



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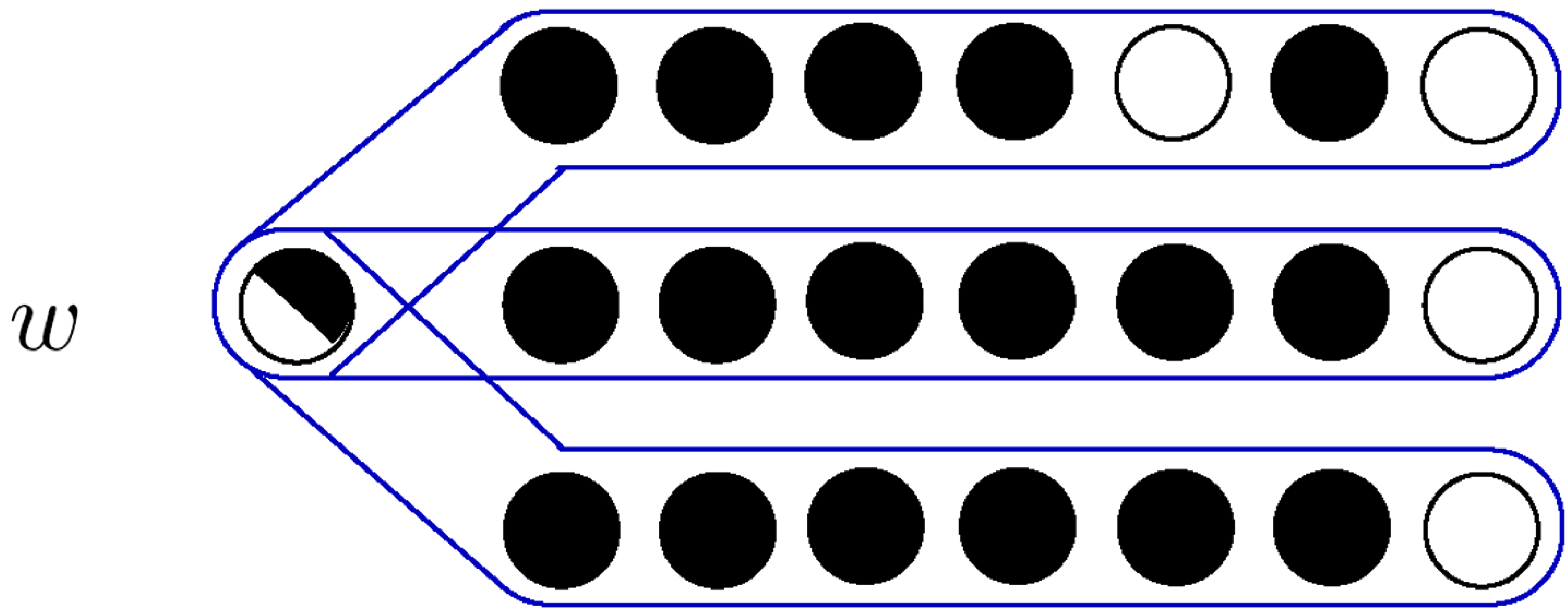

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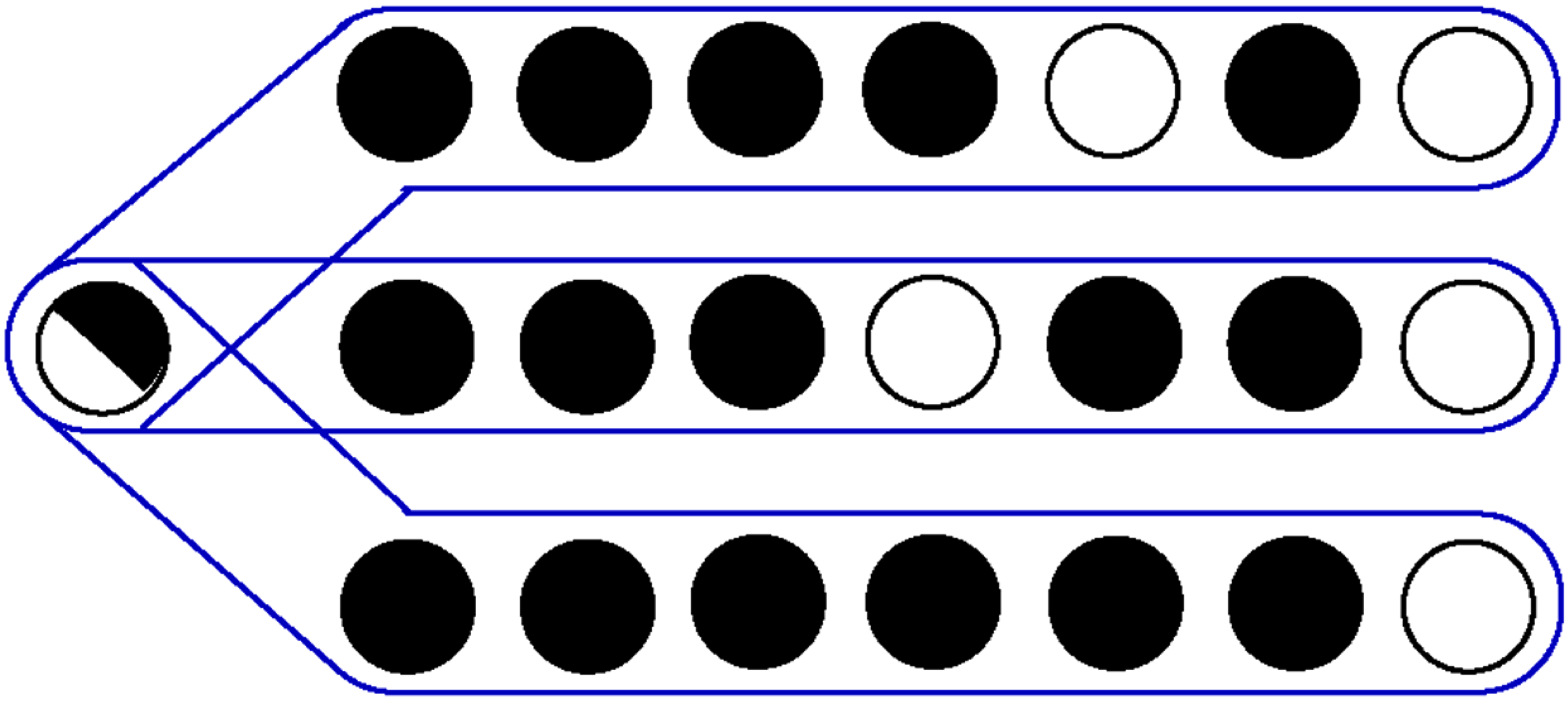

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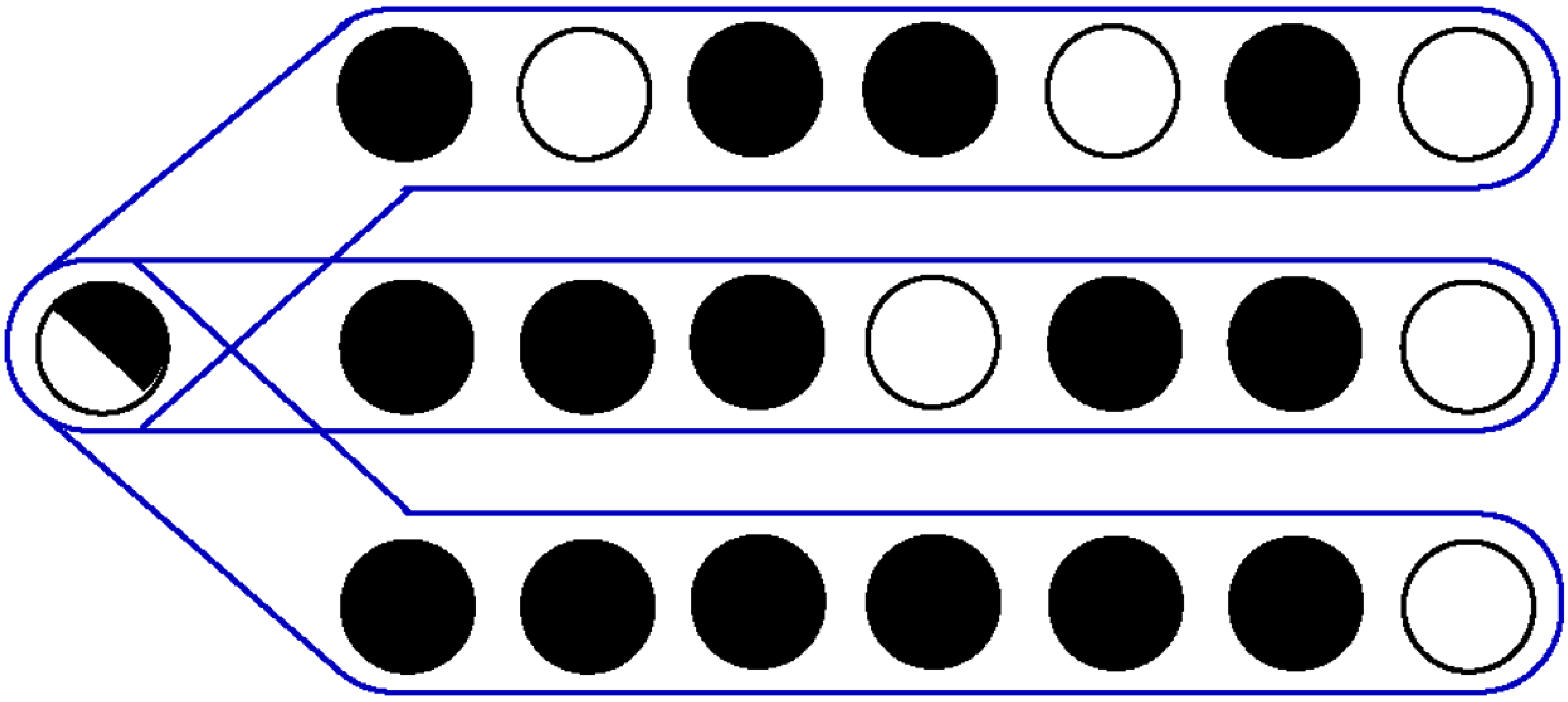

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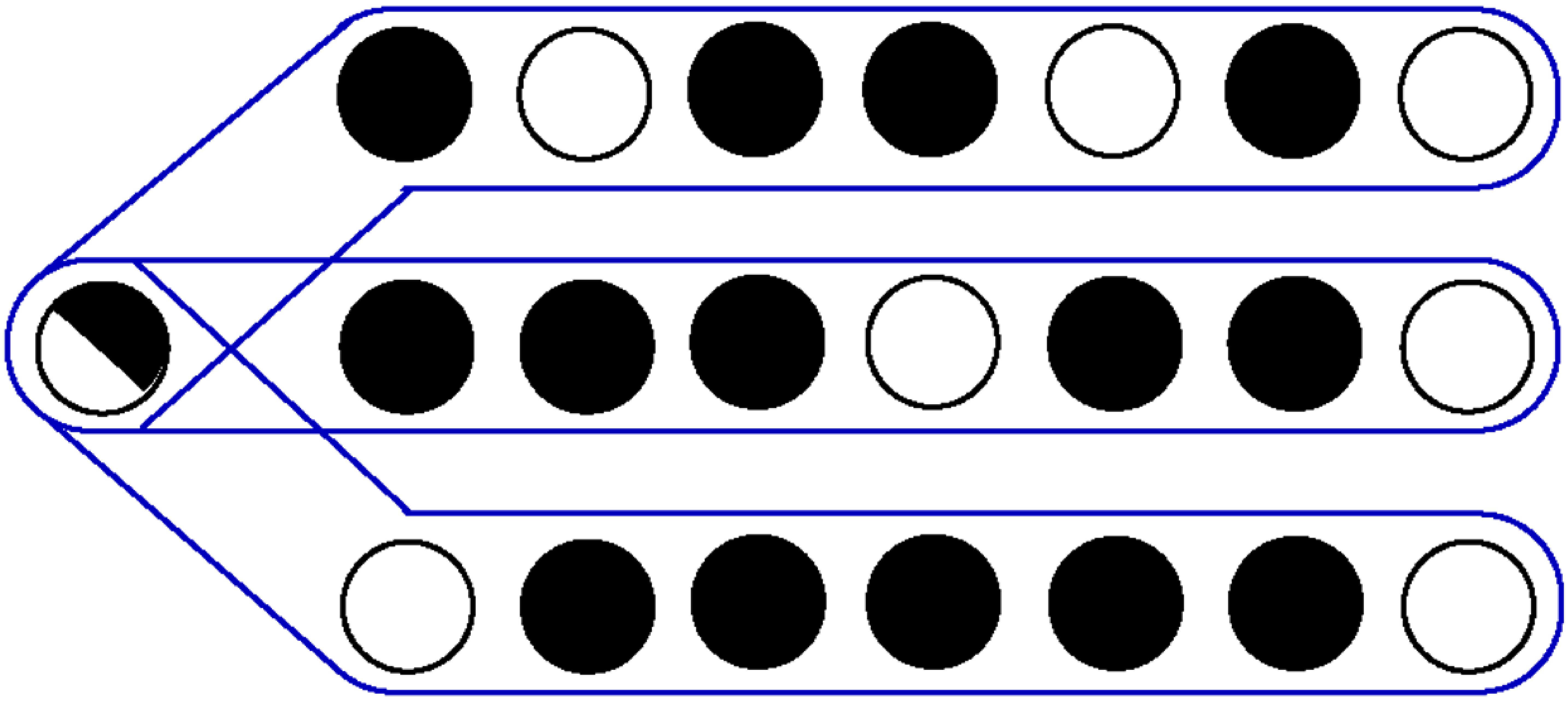

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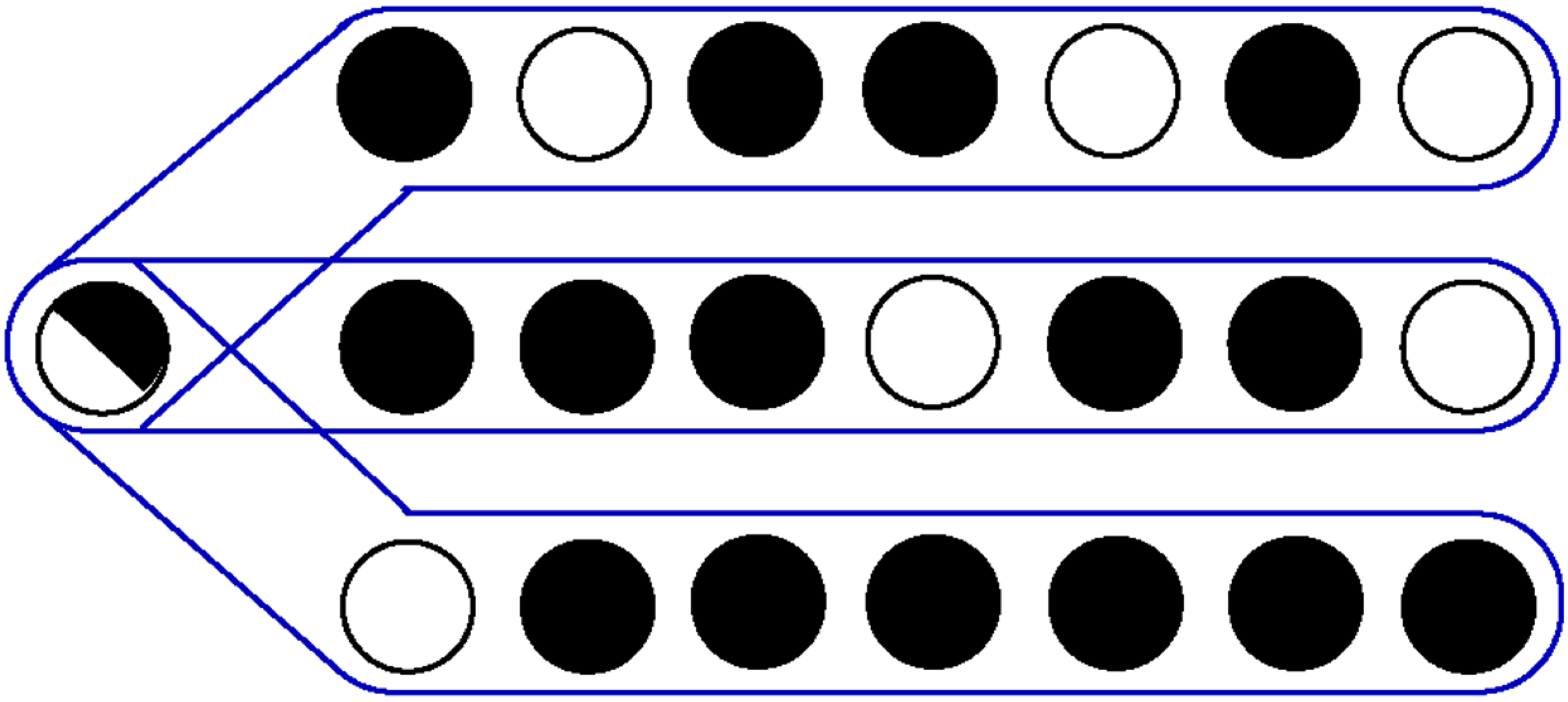

DISTANCE-CHANGE STOPPING TIME ANALYSIS FOR PATH COUPLING (AND A CHOICE OF RIGHT METRIC).

GIVEN $(I_t, J_t) \in S$, A STOPPING TIME T_t FOR (I_t, J_t) IS THE SMALLEST $t' > t$ S.T. $d(I_{t'}, J_{t'}) \neq d(I_t, J_t)$.

LET

$$\mathbb{E}[d(I_{T_t}, J_{T_t})] \leq \alpha\, d(I_t, J_t)$$

FOR **ALL** $(I_t, J_t) \in S$.

IF $\alpha < 1$, WE **DEFINE** A NEW "CONTRACTION" METRIC d',

$$d'(I_t, J_t) = (1-\alpha)\, d(I_t, J_t) + \mathbb{E}[d(I_{T_t}, J_{T_t})] \; (\leq d(I_t, J_t))$$

THE NEW **METRIC** d' ALLOWS US TO WORK WITH THE STANDARD <u>ONE-STEP</u> **PATH COUPLING**, AVOIDING <u>MULTISTEP</u> ANALYSIS.

<u>THEOREM</u> (BORDEWICH, DYER, K. '06)

<u>SUPPOSE</u> FOR ALL $(I_0, J_0) \in S$

(1) $\Pr[T_0 \leq k] \geq p > 0$,

(2) $\mathbb{E}[d(I_{T_0}, J_{T_0})] \leq \alpha < 1$.

<u>THEN</u>

$$\tau(\varepsilon) \leq \frac{k(2-\alpha)}{p(1-\alpha)} \ln\left(\frac{2eD}{\varepsilon}\right)$$

FOR AN (m, Δ)-
HYPERGRAPH:

$$\mathbb{E}[H(I_{T_0}, J_{T_0})] < 2\Delta/(m-1).$$

THUS WE OBTAIN
RAPID MIXING WHEN

$$2\Delta/(m-1) \leqslant 1$$

OR

WHEN $\underbrace{m \geqslant 2\Delta + 1}$.

RECENT IMPROVEMENT:

THEOREM (BORDEWICH, DYER, K. '06)

LET Δ BE FIXED AND H BE A HYPERGRAPH WITH $m \geq \Delta + 2 \geq 5$ OR $\Delta = 3$ AND $m \geq 2$. THEN THE MARKOV CHAIN $\mathcal{M}(H)$ HAS MIXING TIME $O(n \log n)$.

FURTHER APPLICATIONS:

COUNTING INDEPENDENT SETS WITH FUGACITY λ (HARD-CORE MODEL)

$$\underset{\substack{\uparrow \\ \text{PARTITION FNCT.}}}{Z} = Z(\lambda) = \underbrace{\sum_{I \in \Omega} \lambda^{|I|}}.$$

IF $\lambda > 694/\Delta$, THERE IS NO FPRAS FOR $Z(\lambda)$ (BDK'05).

ADJUST THE PROB.'S FOR "DELETE" AND "INSERT" PHASES OF $\mathcal{M}$ TO $1/_{1+\lambda}$ AND $\lambda/_{1+\lambda}$.

THEORM. THERE EXISTS AN FPRAS FOR HYPERGRAPHS SUCH THAT $m \geq 2\lambda\Delta + 1$.
(THE ADJUSTED MARKOV CHAIN MIXES IN $O(n \log n)$ TIME).

- **COUNTING** <u>HYPERGRAPH COLORINGS</u> (ANTI-FERROMAGNETIC SYSTEMS, EXTENDED POTTS MODELS, ...).

$\mathcal{M}$ <u>SIMULATES</u> GLAUBNER DYNAMICS (HEAT-BATH) <u>FOR</u> q-COLORS.

<u>THEOREM</u>. FOR $m \geq 4$, $q > \Delta$, $\mathcal{M}$ <u>MIXES</u> IN <u>TIME</u> $O(n \log n)$.

FOR $m=3$, $\mathcal{M}$ MIXES IN TIME $O(n \log n)$ FOR $q \geq \lceil \frac{3}{2}\Delta + 1 \rceil$.

(BDK '06)

INTERESTINGLY, THE BOUND FOR $m=2$ (GRAPHS) IS

$q \geq \frac{11}{6}\Delta$ (VIGODA '2000).

STILL ON GRAPHS:

RAPID MIXING FOR INDEPENDENT SETS FOR $\Delta \leq 4$ (LUBY, VIGODA, 1999; DYER, GREENHILL, 2000)

RECENT DETERMINISTIC FPAS FOR $\Delta \leq 5$, (WEITZ, 2006).

// LOWER BOUND:
NO FPRAS FOR INDEPEND. SETS FOR $\Delta \geq 25$.
(DYER, FRIEZE, JERRUM '99)

LOWER APPROX. BOUNDS:

THERE IS NO FPRAS FOR HYPERGRAPH COLORING FOR q SATISFYING

$$2 < q \leq \left(1 - \frac{1}{m}\right)\Delta^{1/(m-1)}$$

FOR $m = 3$ THAT MEANS VERY WEAK BOUND FOR q,

$$q \leq \frac{2}{3}\sqrt{\Delta}.$$

FURTHER RESEARCH:

- IMPROVE THE BOUNDS ON MIXING TIMES (GETTING CLOSER TO THE m, Δ DIAGONAL).
- HOW ABOUT SHARPER LOWER BOUNDS? CERTAINLY NEW METHODS ARE NEEDED ...

■