Tight Approximation Bounds for Vertex Cover on Dense k-Partite Hypergraphs

Marek Karpinski* Richard Schmied† Claus Viehmann‡

Abstract

We establish almost tight upper and lower approximation bounds for the Vertex Cover problem on dense k-partite hypergraphs.

1 Introduction

A hypergraph $H = (V, E)$ consists of a vertex set V and a collection of hyperedges E where a hyperedge is a subset of V. H is called k-uniform if every edge in E contains exactly k vertices. A subset C of V is a vertex cover of H if every edge $e \in E$ contains at least a vertex of C.

The *Vertex Cover* problem in a k-uniform hypergraph H is the problem of computing a minimum cardinality vertex cover in H. It is well known that the problem is NP-hard even for $k = 2$ (cf. [13]). On the other hand, the simple greedy heuristic which chooses a maximal set of nonintersecting edges, and then outputs all vertices in those edges, gives a k-approximation algorithm for the Vertex Cover problem restricted to k-uniform hypergraphs. The best known approximation algorithm achieves a slightly better approximation ratio of $(1 - o(1))k$ and is due to Halperin [11].

On the intractability side, Trevisan [22] provided one of the first inapproximability results for the k-uniform vertex cover problem and obtained a inapproximability factor of $k^{\frac{1}{19}}$ assuming $P \neq NP$. In 2002, Holmerin [11] improved the factor to $k^{1-\epsilon}$. Dinur et al. [7, 8] gave consecutively two lower bounds, first $(k - 3 - \epsilon)$ and later on $(k - 1 - \epsilon)$. Moreover, assuming Khot's Unique Games Conjecture (UGC) [17], Khot and Regev [18] proved an inapproximability factor of $k - \epsilon$ for the Vertex Cover problem on k-uniform hypergraphs. Therefore, it implies that the currently achieved ratios are the best possible.

*Dept. of Computer Science and the Hausdorff Center for Mathematics, University of Bonn. Supported in part by DFG grants and the Hausdorff Center grant EXC59-1. Email: marek@cs.uni-bonn.de

†Dept. of Computer Science, University of Bonn. Work supported by Hausdorff Doctoral Fellowship. Email: schmied@cs.uni-bonn.de

‡Dept. of Computer Science, University of Bonn. Work partially supported by Hausdorff Center for Mathematics, Bonn. Email: viehmann@cs.uni-bonn.de

The Vertex Cover problem restricted to k-partite k-uniform hypergraphs, when the underlying partition is given, was studied by Lovász [20] who achieved a $\frac{k}{2}$-approximation. This approximation upper bound is obtained by rounding the natural LP relaxation of the problem. The above bound on the integrality gap was shown to be tight in [1]. As for the lower bounds, Guruswami and Saket [10] proved that it is NP-hard to approximate the Vertex Cover problem on k-partite k-uniform hypergraphs to within a factor of $\frac{k}{4} - \epsilon$ for $k \geq 5$. Assuming the Unique Games Conjecture, they also provided an inapproximability factor of $\frac{k}{2} - \epsilon$ for $k \geq 3$. More recently, Sachdeva and Saket [21] claimed a nearly optimal NP-hardness factor.

To gain better insights on lower bounds, dense instances of many optimization problems has been intensively studied [2, 15, 16, 14]. The Vertex Cover problem has been investigated in the case of dense graphs, where the number of edges is within a constant factor of n^2, by Karpinski and Zelikovsky [16], Eremeev [9], Clementi and Trevisan [6], later by Bar-Yehuda and Kehat [4] as well as Imamura and Iwama [12].

The Vertex Cover problem restricted to dense balanced k-partite k-uniform hypergraphs was introduced and studied in [5], where it was proved that this restricted version of the problem admits an approximation ratio better than $\frac{k}{2}$ if the given hypergraph is dense enough.

In this paper, we give a new approximation algorithm for the Vertex Cover problem restricted to dense k-partite k-uniform hypergraphs and prove that the achieved approximation ratio is almost tight assuming the Unique Games Conjecture.

2 Definitions and Notations

Given a natural number $i \in \mathbb{N}$, we introduce for notational simplicity the set $[i] = \{1, .., i\}$ and set $[0] = \emptyset$. Let S be a finite set with cardinality s and $k \in [s]$. We will use the abbreviation $\binom{S}{k} = \{S' \subseteq S \mid |S'| = k\}$.
A *k-uniform hypergraph* $H = (V(H), E(H))$ consists of a set of vertices V and a collection $E \subseteq \binom{V}{k}$ of edges. For a k-uniform hypergraph H and a vertex $v \in V(H)$, we define the *neighborhood* $N_H(v)$ of v by $\left(\bigcup_{e \in \{e \in E | v \in e\}} e\right) \setminus \{v\}$ and the *degree* $d_H(v)$ of v to be $|\{e \in E \mid v \in e\}|$. We extend this notion to subsets of $V(H)$, where $S \subseteq V(H)$ obtains the degree $d_H(S)$ by $|\{e \in E \mid S \subseteq e\}|$.
A *k-partite* k-uniform hypergraph $H = (V_1, .., V_k, E(H))$ is a k-uniform hypergraph such that V is a disjoint union of $V_1, .., V_k$ with $|V_i \cap e| = 1$ for every $e \in E$ and $i \in [k]$. In the remainder, we assume that $|V_i| \geq |V_{i+1}|$ for all $i \in [k-1]$ and $k = O(1)$.
A *balanced* k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$ is a k-partite k-uniform hypergraph with $|V_i| = \frac{|V|}{k}$ for all $i \in [k]$. We set $n = |V|$ and $m = |E|$ as usual.
For a k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$ and $v \in V_k$, we introduce the *v-induced hypergraph* $H(v)$, where the edge set of $H(v)$ is defined by

$\{e \setminus \{v\} \mid v \in e \in E(H)\}$ and the vertex set of $H(v)$ is partitioned into $V_i \cap N_H(v)$ with $i \in [k-1]$.

A *vertex cover* of a k-uniform hypergraph $H = (V(H), E(H))$ is a subset C of $V(H)$ with the property that $e \cap C \neq \emptyset$ holds for all $e \in E(H)$. The Vertex Cover problem consists of finding a vertex cover of minimum size in a given k-uniform hypergraph. The Vertex Cover problem in k-partite k-uniform hypergraphs is the restricted problem, where a k-partite k-uniform hypergraph and its vertex partition is given as a part of the input.

We define a k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$ as ϵ-*dense* for an $\epsilon \in [0,1]$ if the following condition holds:

$$|E(H)| \quad \geq \quad \epsilon \prod_{i \in [k]} |V_i|$$

For $\ell \in [k-1]$, we introduce the notion of ℓ-wise ϵ-dense k-partite k-uniform hypergraphs. Given a k-partite k-uniform hypergraph H, if there exists an $I \in \binom{[k]}{\ell}$ and an $\epsilon \in [0,1]$ such that for all S with the property $|V_i \cap S| = 1$ for all $i \in I$ the condition

$$d_H(S) \quad \geq \quad \epsilon \prod_{i \in [k] \setminus I} |V_i|$$

holds, we define H to be ℓ-wise ϵ-dense.

3 Our Results

In this paper, we give an improved approximation upper bound for the Vertex Cover problem restricted to ϵ-dense k-partite k-uniform hypergraphs. The approximation algorithm in [5] yields an approximation ratio of

$$\frac{k}{k - (k-2)(1-\epsilon)^{\frac{1}{k-\ell}}}$$

for ℓ-wise ϵ-dense balanced k-partite k-uniform hypergraphs. Here, we design an algorithm with an approximation factor of

$$\frac{k}{2 + (k-2)\epsilon}$$

for the ϵ-dense case which also improves on the ℓ-wise ϵ-dense balanced case for all $\ell \in [k-2]$ and matches their bound when $\ell = k-1$. A further advantage of this algorithm is that it applies to a larger class of hypergraphs since the considered hypergraph is not necessarily required to be balanced.

As a byproduct, we obtain a constructive proof that a vertex cover of an ϵ-dense k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$ is bounded from below by $\epsilon|V_k|$, which is shown to be sharp by constructing a family of tight examples.

On the other hand, we provide inapproximability results for the Vertex Cover problem restricted to ℓ-wise ϵ-dense balanced k-partite k-uniform hypergraphs under the Unique Games Conjecture. We also prove that this reduction yields a matching lower bound if we use a conjecture on the Unique Games hardness of the Vertex Cover problem restricted to balanced k-partite k-uniform hypergraphs. This means that further restrictions such as ℓ-wise density cannot lead to improved approximation ratios and our proposed approximation algorithm is best possible assuming this conjecture. In addition, we are able to prove an inapproximability factor under $P \neq NP$.

4 Approximation Algorithm

In this section, we give a polynomial time approximation algorithm with improved approximation factor for the Vertex Cover problem restricted to ϵ-dense k-partite k-uniform hypergraphs.

We state now our main result.

Theorem 1. *There exists a polynomial time approximation algorithm with approximation ratio*

$$\frac{k}{2+(k-2)\epsilon}$$

for the Vertex Cover problem in ϵ-dense k-partite k-uniform hypergraphs.

A crucial ingredient of the proof of Theorem 1 is Lemma 1, in which we show that we can extract efficiently a large part of an optimal vertex cover of a given ϵ-dense k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$. More precisely, we obtain in this way a constructive proof that the size of a vertex cover of H is bounded from below by $\epsilon|V_k|$. The procedure for the extraction of a part of an optimal vertex cover is given in Figure 1.

We now formulate Lemma 1:

Lemma 1. *Let $H = (V_1, .., V_k, E(H))$ be an ϵ-dense k-partite k-uniform hypergraph with $k \geq 1$. Then, the procedure $Extract(\cdot)$ computes in polynomial time a collection R of subsets of $V(H)$ such that the size of R is polynomial in $|V(H)|$ and R contains a set S, which is a subset of an optimal vertex cover of H and its cardinality is at least $\epsilon|V_k|$.*

As a consequence, we obtain directly:

Corollary 1. *Given an ϵ-dense k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$ with $k \geq 1$, the cardinality of an optimal vertex cover of H is bounded from below by $\epsilon|V_k|$.*

Before we prove Lemma 1, we describe the main idea of the proof. Let OPT denote an optimal vertex cover of H. The procedure $Extract(\cdot)$ tests for the set $R = \{v_1, .., v_p\}$ of the p heaviest vertices of V_k, if $\{v_1, .., v_{u-1}\} \subseteq OPT$ and $v_u \notin OPT$ for every $u \in [p]$. Clearly, either $R \subseteq OPT$ or there exists a v_u such that $v_u \notin OPT$. If the procedure already possesses a part of OPT denoted by R_u, then,

Procedure $Extract(\cdot)$

Input: ϵ-dense k-partite k-uniform hypergraph $H = (V_1, .., V_k, E)$ with $k \geq 1$

1. IF $k = 1$ THEN
   (a) RETURN $\{\bigcup_{e \in E} e\}$
2. ELSE:
   (a) Let $(v_1, .., v_p)$ be the vector consisting of the first $p = \left\lceil \frac{|E|}{\prod_{l \in [k-1]} |V_l|} \right\rceil$ heaviest vertices of V_k with $d_H(v_i) \geq d_H(v_{i+1})$
   (b) $R = \{\{v_1, .., v_p\}\}$
   (c) FOR $i = 1, .., p$ DO:
      i. $R_i = \{v_k \mid k \in [i-1]\}$
      ii. Invoke $Extract(H(v_i))$ with output O
      iii. $R = R \cup \{R_i \cup S \mid S \in O\}$
3. RETURN R

Figure 1: Procedure $Extract$

$Extract(\cdot)$ tries to obtain a large part of an optimal vertex cover of the v_u-induced hypergraph $H(v_u)$. Hence, we have to show that $H(v_u)$ must still be dense enough. We now give the proof of Lemma 1.

Proof. The proof of Lemma 1 will be split in several parts. In particular, we show that given an ϵ-dense k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$, the procedure $Extract(\cdot)$ and its output R possess the following properties:

1. $Extract(\cdot)$ constructs R in polynomial time and the cardinality of R is $O(n^k)$.
2. There is a $S \in R$ such that S is a subset of an optimal vertex cover of H.
3. For every $S \in R$, the cardinality of S is at least $|S| \geq \epsilon |V_k|$.

(1.) Clearly, R is upper bounded by $|V_1|^k = O(n^k)$ and therefore, the running time of $Extract(\cdot)$ is $O(n^k)$.
(2.) and (3.) We prove the remaining properties by induction. If we have $k = 1$, the set $\bigcup_{e \in E(H)} e$ is by definition an optimal vertex cover of $H = (V_1, E(H))$. Since H is ϵ-dense, the cardinality of $|E(H)|$ is lower bounded by $\epsilon |V_1|$.
We assume that $k > 1$. Let $H = (V_1, .., V_k, E(H))$ be an ϵ-dense k-partite k-uniform hypergraph and $OPT \subseteq V(H)$ an optimal vertex cover of H. Let $(v_1, .., v_p)$ be the vector consisting of the first $p = \left\lceil \frac{|E(H)|}{\prod_{l \in [k-1]} |V_l|} \right\rceil$ heaviest vertices of V_k with $d_H(v_i) \geq d_H(v_{i+1})$. If $\{v_1, .., v_p\}$ is contained in OPT, we have constructed a subset of an

optimal vertex cover with cardinality

$$p = \left\lceil \frac{|E(H)|}{\prod\limits_{l \in [k-1]} |V_l|} \right\rceil \quad \geq \quad \frac{\epsilon \prod\limits_{l \in [k]} |V_l|}{\prod\limits_{l \in [k-1]} |V_l|} \quad \geq \quad \epsilon |V_k|.$$

Otherwise, there is an $u \in [p]$ such that $R_u \subseteq OPT$ and $v_u \notin OPT$. But this means that an optimal vertex cover of H contains an optimal vertex cover of the v_u-induced $(k-1)$-partite $(k-1)$-uniform hypergraph $H(v_u)$ in order to cover the edges $e \in \{e \in E \mid v_u \in e\}$. The situation is depicted in Figure 2.

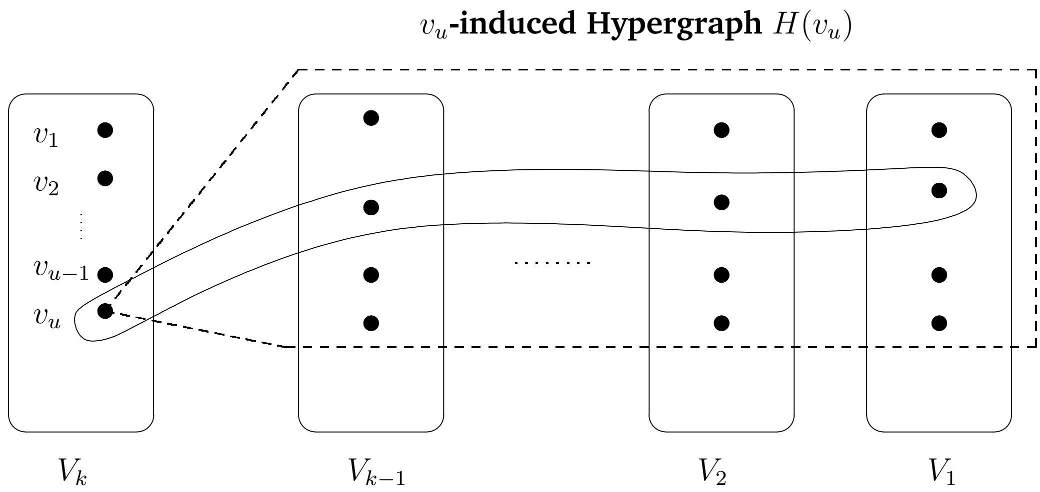

Figure 2: The v_u-induced $(k-1)$-partite $(k-1)$-uniform hypergraph $H(v_u)$

By our induction hypothesis, $Extract(H(v_u))$ contains a set S_u which is a subset of a minimum vertex cover of $H(v_u)$ and of OPT. The only claim, which remains to be proven, is that the cardinality of S_u is large enough. More precisely, we show that $|S_u|$ can be lower bounded by $\epsilon|V_k| - |R_u|$. Therefore, we need to analyze the density of the v_u-induced hypergraph $H(v_u)$. The edge set of $H(v_u)$ is given by $\{e \setminus \{v_u\} \mid v_u \in e \in E\}$. Thus, we have to obtain a lower bound on the degree of v_u. Since $|\{e \in E \mid e \cap R_u \neq \emptyset\}|$ is upper bounded by $|R_u| \prod_{l \in [k-1]} |V_l|$, the vertices in $V_k \setminus R_u$ possess the average degree of at least

$$\frac{\sum\limits_{v \in V_k \setminus R_u} deg_H(v)}{|V_k \setminus R_u|} \geq \frac{\epsilon \prod\limits_{l \in [k]} |V_l| - |\{e \in E \mid e \cap R_u \neq \emptyset\}|}{|V_k \setminus R_u|} \tag{1}$$

$$\geq \frac{\epsilon \prod\limits_{l \in [k]} |V_l| - |R_u| \prod\limits_{l \in [k-1]} |V_l|}{|V_k \setminus R_u|} \tag{2}$$

$$\geq \frac{(\epsilon |V_k| - |R_u|) \prod\limits_{l \in [k-1]} |V_l|}{|V_k \setminus R_u|} \tag{3}$$

Since the heaviest vertex in $V_k \setminus R_u$ must have a degree of at least $\frac{(\epsilon|V_k|-|R_u|)\prod_{l\in[k-1]}|V_l|}{|V_k\setminus R_u|}$, we deduce that the edge set of $H(v_u)$ denoted by E_u can be lower bounded by

$$|E_u| \quad \geq \quad \frac{(\epsilon|V_k| - |R_u|) \prod\limits_{l\in[k-1]} |V_l|}{|V_k \setminus R_u|}$$

Let $H(v_u)$ be defined by $(V_1^u, .., V_{k-1}^u, E_u)$ with $|V_i^u| \leq |V_i|$ for all $i \in [k-1]$. By our induction hypothesis, the size of every set contained in $Extract(\cdot)$ is at least

$$\frac{|E_u|}{\prod\limits_{l\in[k-1]} |V_l^u|}|V_{k-1}| \quad \geq \quad \frac{(\epsilon|V_k| - |R_u|) \prod\limits_{l\in[k-1]} |V_l|}{|V_k \setminus R_u| \prod\limits_{l\in[k-1]} |V_l^u|}|V_{k-1}| \tag{4}$$

$$\geq \quad \frac{(\epsilon|V_k| - |R_u|) \prod\limits_{l\in[k-1]} |V_l|}{|V_k \setminus R_u| \prod\limits_{l\in[k-1]} |V_l|}|V_{k-1}| \tag{5}$$

$$\geq \quad \frac{(\epsilon|V_k| - |R_u|)}{|V_k \setminus R_u|}|V_k| \tag{6}$$

$$\geq \quad \frac{(\epsilon|V_k| - |R_u|)}{|V_k|}|V_k| \; = \; \epsilon|V_k| - |R_u| \tag{7}$$

In (4), we used the fact that $|V_i^u| \leq |V_i|$ for all $i \in [k-1]$. Whereas in (5), we used our assumption $|V_k| \leq |V_{k-1}|$. All in all, we obtain

$$|R_u \cup S_u| \quad \geq \quad |R_u| + (|\epsilon|V_k| - |R_u|) \; = \; \epsilon|V_k|. \tag{8}$$

Clearly, this argumentation on the size of $R_u \cup S_u$ holds for every $u \in [p]$ and the proof of Lemma 1 follows. □

Before we state our approximation algorithm and prove Theorem 1, we show that the bound in Lemma 1 is tight. In particular, we define a family of ϵ-dense k-partite k-uniform hypergraphs $H(k, l, \epsilon) = (V_1, .., V_k, E(H_l))$ with $|V_i| = \frac{|V|}{k}$ for all $i \in [k]$, $k \geq 1$, $\epsilon \in \{\frac{u}{l} \mid u \in [l]\}$ and $l \geq 1$ such that $Extract(\cdot)$ returns a subset of an optimal vertex cover with cardinality of exactly $\epsilon|V_k|$.

Lemma 2. *The bound of Lemma 1 is tight.*

Proof. Let us define $H(k, p, \epsilon) = (V_1, .., V_k, E)$. For a fixed $p \geq 1$ and $k \geq 1$, every partition V_i with $i \in [k]$ consists of a set of l vertices. Let us fix a $\epsilon = \frac{u}{l}$ with $u \in [l]$. Then, $H(k, l, \epsilon)$ contains the set $V_k^u \subseteq V_k$ of u vertices such that $E = \{\{v_1, v_2, .., v_k\} \mid v_1 \in V_k^u, v_2 \in V_2, .., v_k \in V_k\}$. An example of such a hypergraph is depicted in Figure 3.

Notice that $H(k, l, \epsilon) = (V_1, .., V_k, E)$ is ϵ-dense, since

$$\frac{|E|}{\prod\limits_{j\in[k]} |V_j|} = \frac{|V_k^u|}{|V_k|} = \frac{u}{l} = \epsilon.$$

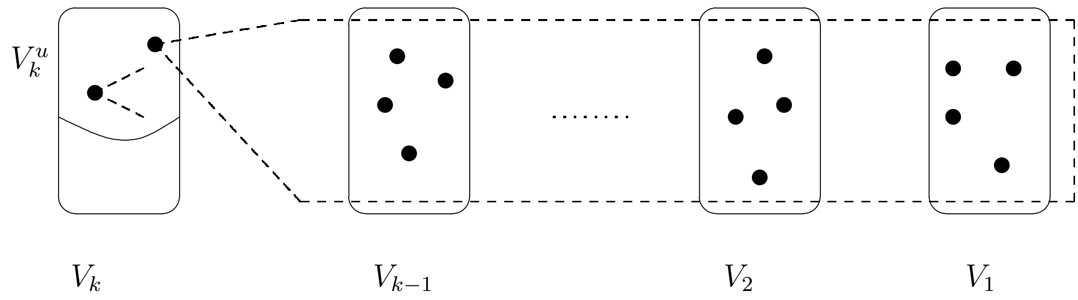

Figure 3: An example of a hypergraph $H(k,l,\epsilon)$

The procedure $Extract(\cdot)$ returns a set R, in which V_k^u is contained, since V_k^u is the set of the p heaviest vertices of V_k. Hence, we obtain $|V_k^u| = \frac{|V_k^u|}{|V_k|}|V_k| = \epsilon|V_k|$. On the other hand, the remaining hypergraph $H' = (V_1,..,V_k \setminus V_k^u, E(H'))$ with edge set $E(H') = \{e \in E \mid e \cap V_k^u = \emptyset\}$ is already covered, since $E(H')$ is by definition of $H(k,p,\epsilon)$ the empty set. Therefore, V_k^u is a vertex cover of $H(k,p,\epsilon)$ and since, according to Corollary 1, every vertex cover is bounded from below by $\epsilon|V_k|$, V_k^u must be an optimal vertex cover. $\square$

Next, we state our approximation algorithm for the Vertex Cover problem in ϵ-dense k-partite k-uniform hypergraphs defined in Figure 4. The approximation algorithm combines the procedure $Extract(\cdot)$ to generate a large enough subset of an optimal vertex cover together with the $\frac{k}{2}$-approximation algorithm due to Lovász [20] applied to the remaining instance.

Algorithm $Approx(\cdot)$

Input: ϵ-dense k-partite k-uniform hypergraph $H = (V_1,..,V_k,E)$ with $k \geq 3$

1. $T = \{V_k\}$
2. invoke procedure $Extract(H)$ with output R
3. for all $S \in R$ do :
   (a) $H_S = (V(H) \setminus S, \{e \in E(H) \mid e \cap S = \emptyset\})$
   (b) obtain a $(\frac{k}{2})$-approximate solution S_k for H_S
   (c) $T = T \cup \{S_k \cup S\}$
4. Return the smallest set in T

Figure 4: Algorithm $Approx(\cdot)$

We now prove Theorem 1.

Proof. Let $H = (V_1, .., V_k, E)$ be an ϵ-dense k-partite k-uniform hypergraph. From Lemma 1, we know that the procedure $Extract(\cdot)$ returns in polynomial time a collection C of subsets of $V(H)$ such that there is a set S in C, which is contained in an optimal vertex cover of H. Moreover, we know that the size of S is lower bounded by $\epsilon|V_k|$.
Next, we analyze the approximation ratio of our approximation algorithm $Approx(\cdot)$. Clearly, the size of an optimal vertex cover of H is upper bounded by $|V_k|$. Let us denote by OPT' the size of an optimal vertex cover of the remaining hypergraph H' defined by removing all edges e of H with $e \cap S \neq \emptyset$. Furthermore, let S' be the solution of the $\frac{k}{2}$-approximation algorithm applied to H'. The approximation ratio of $Approx(\cdot)$ is bounded by

$$\frac{|S| + |S'|}{|S| + |OPT'|} \leq \frac{|S| + \frac{k}{2}|OPT'|}{|S| + |OPT'|} \leq \frac{k}{\frac{k|S|+k|OPT'|}{|S|+\frac{k}{2}|OPT'|}} \tag{9}$$

$$\leq \frac{k}{\frac{2|S|+(k-2)|S|+k|OPT'|}{|S|+\frac{k}{2}|OPT'|}} \tag{10}$$

$$\leq \frac{k}{2 + (k-2)\frac{|S|}{|S|+\frac{k}{2}|OPT'|}} \tag{11}$$

$$\leq \frac{k}{2 + (k-2)\frac{|S|}{|V_k|}} \tag{12}$$

$$\leq \frac{k}{2 + (k-2)\frac{\epsilon|V_k|}{|V_k|}} \tag{13}$$

$$\leq \frac{k}{2 + (k-2)\epsilon} \tag{14}$$

In (11), we used the fact that the size of the output of $Approx(\cdot)$ is upper bounded by $|V_k|$. Therefore, we have $|S| + \frac{k}{2}|OPT'| \leq |V_k|$. In (12), we know from Lemma 1 that $|S| \geq \epsilon|V_k|$. □

5 Inapproximability Results

In this section, we prove hardness results for the Vertex Cover problem restricted to ℓ-wise ϵ-dense balanced k-uniform k-partite hypergraphs under the Unique Games Conjecture [17] as well as under the assumption $P \neq NP$.

5.1 UGC-Hardness

The Unique Games-hardness result of [10] was obtained by applying the result of Kumar et al. [19], with a modification to the LP integrality gap due to Ahorani et al. [1]. More precisely, they proved the following inapproximability result:

Theorem 2. *[10] For every $\delta > 0$ and $k \geq 3$, there exist a n_δ such that given $H = (V_1, .., V_k, E(H))$ as an instance of the Vertex Cover problem in balanced k-partite k-uniform hypergraphs with $|V(H)| \geq n_\delta$, the following is UGC-hard to decide:*

- *The size of a vertex cover of H is at least $|V| \left(\frac{1}{2(k-1)} - \delta\right)$.*
- *The size of an optimal vertex cover of H is at most $|V| \left(\frac{1}{k(k-1)} + \delta\right)$.*

As the starting point of our reduction, we use Theorem 2 and prove the following:

Theorem 3. *For every $\delta > 0$, $\epsilon \in (0, 1)$, $\ell \in [k-1]$, and $k \geq 3$, there exists no polynomial time approximation algorithm with an approximation ratio*

$$\frac{k}{2 + \frac{2(k-1)(k-2)\epsilon}{k+(k-2)\epsilon}} - \delta$$

for the Vertex Cover problem in ℓ-wise ϵ-dense k-partite k-uniform hypergraphs assuming the Unique Games Conjecture.

Proof. First, we concentrate on the ϵ-dense case and afterwards, we extend the range of ℓ. As a starting point of the reduction, we use the k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$ from Theorem 2 and construct an ϵ-dense k-partite k-uniform hypergraph $H' = (V_1', .., V_k', E')$.
Let us start with the description of H'. First, we join the set C_i of $\frac{\epsilon}{1-\epsilon}\frac{n}{k}$ vertices to V_i for every $i \in [k]$ and add all possible edges e of H' to E' with the restriction $C_1 \cap e \neq \emptyset$. Thus, we obtain $|V_i'| = \frac{n}{k} + \frac{\epsilon}{1-\epsilon}\frac{n}{k}$ for all $i \in [k]$.
Now, let us analyze how the size of the optimal solution of H' transforms. We denote by OPT' an optimal vertex cover of H'. The UGC-hard decision question from Theorem 2 transforms into the following:

$$n\left(\frac{1}{2(k-1)} - \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k} \leq |OPT'| \text{ or } |OPT'| \leq n\left(\frac{1}{k(k-1)} + \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k}$$

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in ϵ-dense hypergraphs for every $\delta' > 0$ to within:

$$\frac{n\left(\frac{1}{2(k-1)} - \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k}}{n\left(\frac{1}{k(k-1)} + \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k}} = \frac{\frac{1-\epsilon}{2(k-1)} - \delta(1-\epsilon) + \frac{\epsilon}{k}}{\frac{1-\epsilon}{k(k-1)} + \delta(1-\epsilon) + \frac{\epsilon}{k}} \tag{15}$$

$$= \frac{\frac{(1-\epsilon)k}{2(k-1)k} + \frac{2\epsilon(k-1)}{2k(k-1)}}{\frac{1-\epsilon}{(k-1)k} + \frac{\epsilon(k-1)}{k(k-1)}} - \delta' \tag{16}$$

$$\frac{\frac{(1-\epsilon)k}{2(k-1)k}+\frac{2\epsilon(k-1)}{2k(k-1)}}{\frac{1-\epsilon}{(k-1)k}+\frac{\epsilon(k-1)}{k(k-1)}}-\delta' = \frac{\frac{k-\epsilon k+2\epsilon k-2\epsilon}{2(k-1)k}}{\frac{1-\epsilon+\epsilon k-\epsilon}{(k-1)k}}-\delta' \tag{17}$$

$$= \frac{k+(k-2)\epsilon}{2(1+(k-2)\epsilon)}-\delta' \tag{18}$$

$$= \frac{k}{\frac{2k(1+(k-2)\epsilon)}{k+(k-2)\epsilon}}-\delta' \tag{19}$$

$$= \frac{k}{\frac{2k+2(k-2)\epsilon+(2k-2)(k-2)\epsilon}{k+(k-2)\epsilon}}-\delta' \tag{20}$$

$$= \frac{k}{2+\frac{(2k-2)(k-2)\epsilon}{k+(k-2)\epsilon}}-\delta' \tag{21}$$

$$= \frac{k}{2+\frac{2(k-1)(k-2)\epsilon}{k+(k-2)\epsilon}}-\delta' \tag{22}$$

Finally, we have to verify that the constructed hypergraph H' is indeed ϵ-dense. Notice that H' can have at most $(|V_1'|)^k = (\frac{n}{k}+\frac{\epsilon}{1-\epsilon}\frac{n}{k})^k$ edges. Therefore, we obtain the following:

$$\frac{\left(\frac{\epsilon}{1-\epsilon}\frac{n}{k}\right)\left(\frac{n}{k}+\frac{\epsilon}{1-\epsilon}\frac{n}{k}\right)^{k-1}}{\left(\frac{n}{k}+\frac{\epsilon}{1-\epsilon}\frac{n}{k}\right)^k} = \frac{\frac{n}{k}\frac{\epsilon}{1-\epsilon}}{\frac{n}{k}\left(1+\frac{\epsilon}{1-\epsilon}\right)} = \frac{\frac{\epsilon}{1-\epsilon}}{\frac{1+\epsilon-\epsilon}{1-\epsilon}} = \epsilon$$

Notice that the constructed hypergraph is also ℓ-wise ϵ-dense balanced. Hence, we obtain the same inapproximability factor in this case as well. □

Next, we combine the former construction with a conjecture about Unique Games hardness of the Vertex Cover problem in balanced k-partite k-uniform hypergraphs. In particular, we postulate the following:

Conjecture 1. *Given a balanced k-partite k-uniform hypergraph $H = (V_1, .., V_k, E(H))$ with $k \geq 3$, let OPT denote an optimal vertex cover of H. For every $\delta > 0$, the following is UGC-hard to decide:*

$$|V|\left(\frac{1}{k}-\delta\right) \leq |OPT| \quad \textit{or} \quad |OPT| \leq |V|\left(\frac{2}{k^2}+\delta\right)$$

Combining Conjecture 1 with the construction in Theorem 3, it yields the following inapproximability result which matches precisely the approximation upper bound achieved by our approximation algorithm described in Section 4:

Theorem 4. *For every $\delta > 0$, $\epsilon \in (0,1)$, $\ell \in [k-1]$, and $k \geq 3$, there exists no polynomial time approximation algorithm with an approximation ratio*

$$\frac{k}{2+(k-2)\epsilon}-\delta$$

for the Vertex Cover problem in ℓ-wise ϵ-dense k-partite k-uniform hypergraphs assuming Conjecture 1.

Proof. The UGC-hard decision question from Conjecture 1 transforms into the following:

$$n\left(\frac{1}{k}-\delta\right)+\frac{\epsilon}{1-\epsilon}\frac{n}{k} \leq |OPT'| \quad \text{or} \quad |OPT'| \leq n\left(\frac{2}{k^2}+\delta\right)+\frac{\epsilon}{1-\epsilon}\frac{n}{k}$$

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in ϵ-dense k-partite k-uniform hypergraphs for every $\delta' > 0$ to within:

$$\frac{n\left(\frac{1}{k}-\delta\right)+\frac{\epsilon}{1-\epsilon}\frac{n}{k}}{n\left(\frac{2}{k^2}+\delta\right)+\frac{\epsilon}{1-\epsilon}\frac{n}{k}} = \frac{n\left(\frac{1}{k}-\delta\right)(1-\epsilon)+\frac{\epsilon n}{k}}{n\left(\frac{2}{k^2}+\delta\right)(1-\epsilon)+\frac{\epsilon n}{k}} \tag{23}$$

$$= \frac{\frac{n}{k}}{n\left(\frac{2}{k^2}\right)(1-\epsilon)+\frac{k\epsilon n}{k^2}}-\delta' \tag{24}$$

$$= \frac{k}{2(1-\epsilon)+k\epsilon}-\delta' \tag{25}$$

$$= \frac{k}{2+(k-2)\epsilon}-\delta' \tag{26}$$

□

5.2 NP-Hardness

Recently, Sachdeva and Saket proved in [21] a nearly optimal NP-hardness of the Vertex Cover problem on balanced k-uniform k-partite hypergraphs. More precisely, they obtained the following inapproximability result:

Theorem 5. *[21] Given a balanced k-partite k-uniform hypergraph $H = (V, E)$ with $k \geq 4$, let OPT denote an optimal vertex cover of H. For every $\delta > 0$, the following is NP-hard to decide:*

$$|V|\left(\frac{k}{2(k+1)(2(k+1)+1)}-\delta\right) \leq |OPT|$$

or

$$|V|\left(\frac{1}{k(2(k+1)+1)}+\delta\right) \geq |OPT|$$

Combining our reduction from Theorem 2 with Theorem 5, we prove the following inapproximability result under the assumption $P \neq NP$:

Theorem 6. *For every $\delta > 0$, $\epsilon \in (0, 1)$, $\ell \in [k-1]$, and $k \geq 4$, there is no polynomial time approximation algorithm with an approximation ratio*

$$\frac{k^2(1-\epsilon)+\epsilon 2(k+1)(2(k+1)+1)}{2(k+1)[1-\epsilon+\epsilon(2(k+1)+1)]}-\delta$$

for the Vertex Cover problem in ℓ-wise ϵ-dense k-partite k-uniform hypergraphs assuming $P \neq NP$.

Proof. The NP-hard decision question from Theorem 5 transforms into the following:

$$n\left(\frac{k}{2(k+1)(2(k+1)+1)} - \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k} \leq |OPT'|$$

or

$$n\left(\frac{1}{k(2(k+1)+1)} + \delta\right) + \frac{\epsilon}{1-\epsilon}\frac{n}{k} \geq |OPT'|$$

Assuming $NP \neq P$, this implies the hardness of approximating the Vertex Cover problem in ϵ-dense hypergraphs for every $\delta' > 0$ to within:

$$\frac{\frac{k(1-\epsilon)}{2(k+1)(2(k+1)+1)} + \frac{\epsilon}{k}}{\frac{1-\epsilon}{k(2(k+1)+1)} + \frac{\epsilon}{k}} - \delta' = \frac{\frac{k^2(1-\epsilon)+\epsilon 2(k+1)(2(k+1)+1)}{k2(k+1)(2(k+1)+1)}}{\frac{1-\epsilon+\epsilon(2(k+1)+1)}{k(2(k+1)+1)}} - \delta' \tag{27}$$

$$= \frac{k^2(1-\epsilon)+\epsilon 2(k+1)(2(k+1)+1)}{2(k+1)[1-\epsilon+\epsilon(2(k+1)+1)]} - \delta' \tag{28}$$

□

6 Further Research

An interesting question remains about even tighter lower approximation bounds for our problem, perhaps connecting it more closely to the integrality gap issue of the LP of Lovász [20].

Acknowledgment

We thank Jean Cardinal for many stimulating discussions.

References

[1] R. Aharoni, R. Holzman, and M. Krivelevich, *On a Theorem of Lovász on Covers in r-Partite Hypergraphs*, Combinatorica 16, pp. 149–174, 1996.

[2] S. Arora, D. Karger, and M. Karpinski, *Polynomial Time Approximation Schemes for Dense Instances of NP-Hard Problems*, Proc. 27th ACM STOC (1995), pp. 284–293; see also in J. Comput. Syst. Sci. 58, pp. 193–210, 1999.

[3] N. Bansal and S. Khot, *Inapproximability of Hypergraph Vertex Cover and Applications to Scheduling Problems*, Proc. 37th ICALP (2010), LNCS 6198, pp. 250–261, 2010.

[4] R. Bar-Yehuda and Z. Kehat, *Approximating the Dense Set-Cover Problem*, J. Comput. Syst. Sci. 69, pp. 547–561, 2004.

[5] J. Cardinal, M. Karpinski, R. Schmied, and C. Viehmann, *Approximability of Vertex Cover in Dense Bipartite Hypergraphs*, Research Report CS-85319, University of Bonn, 2011.

[6] A. Clementi and L. Trevisan, *Improved Non-Approximability Results for Minimum Vertex Cover with Density Constraints*, Theor. Comput. Sci. 225, pp. 113–128, 1999.

[7] I. Dinur, V. Guruswami, and S. Khot *Vertex Cover on k-Uniform Hypergraphs is Hard to Approximate within Factor $(k-3-\epsilon)$*, ECCC TR02-027, 2002.

[8] I. Dinur, V. Guruswami, S. Khot, and O. Regev, *A New Multilayered PCP and the Hardness of Hypergraph Vertex Cover*, SIAM J. Comput. 34, pp. 1129–1146, 2005.

[9] A. Eremeev, *On some Approximation Algorithms for Dense Vertex Cover Problem*, Proc. Symp. on Operations Research (1999), pp. 48–52.

[10] V. Guruswami and R. Saket, *On the Inapproximability of Vertex Cover on k-Partite k-Uniform Hypergraphs*, Proc. 37th ICALP (2010), LNCS 6198, pp. 360–371, 2010.

[11] E. Halperin, *Improved Approximation Algorithms for the Vertex Cover Problem in Graphs and Hypergraphs* SIAM J. Comput. 31, pp. 1608–1623, 2002.

[12] T. Imamura and K. Iwama, *Approximating Vertex Cover on Dense Graphs*, Proc. 16th ACM-SIAM SODA (2005), pp. 582–589.

[13] R. Karp, *On the Computational Complexity of Combinatorial Problems*, Networks 5, pp. 45–68, 1975.

[14] M. Karpinski, *Polynomial Time Approximation Schemes for Some Dense Instances of NP-Hard Optimization Problems*, Algorithmica 30, pp. 386–397, 2001.

[15] M. Karpinski, A. Rucinski, and E. Szymanska, *Computational Complexity of the Perfect Matching Problem in Hypergraphs with Subcritical Density*, Int. J. Found. Comput. Sci. 21, pp. 905–924, 2010.

[16] M. Karpinski and A. Zelikovsky, *Approximating Dense Cases of Covering Problems*, Proc. DIMACS Workshop on Network Design: Connectivity and Facilities Location (1997), pp. 169–178; also published in ECCC TR97-004, 1997.

[17] S. Khot, *On the Power of Unique 2-Prover 1-Round Games*, Proc. 34th ACM STOC (2002), pp. 767–775.

[18] S. Khot and O. Regev, *Vertex Cover Might be Hard to Approximate to within* $2 - \epsilon$, J. Comput. Syst. Sci. 74, pp. 335-349, 2008.

[19] A. Kumar, R. Manokaran, M. Tulsiani, and N. Vishnoi, *On LP-based approximability for strict CSPs*, Proc. 22nd ACM-SIAM SODA (2011), pp. 1560–1573.

[20] L. Lovász, *On Minimax Theorems of Combinatorics*, Doctoral Thesis, Mathematiki Lapok 26, pp. 209–264, 1975.

[21] S. Sachdeva and R. Saket, *Nearly Optimal NP-Hardness of Vertex Cover on* k*-Uniform* k*-Partite Hypergraphs*, CoRR abs/1105.4175, 2011.

[22] L. Trevisan, *Non-Approximability Results for Optimization Problems on Bounded Degree Instances*, Proc. 33rd ACM STOC (2001), pp. 453–461.