On Approximation Complexity of Edge Dominating Set Problem in Dense Graphs

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Abstract

We study the approximation complexity of the *Minimum Edge Dominating Set* problem in everywhere ϵ-dense and average $\bar{\epsilon}$-dense graphs. More precisely, we consider the computational complexity of approximating a generalization of the Minimum Edge Dominating Set problem, the so called *Minimum Subset Edge Dominating Set* problem. As a direct result, we obtain for the special case of the Minimum Edge Dominating Set problem in everywhere ϵ-dense and average $\bar{\epsilon}$-dense graphs by using the techniques of Karpinski and Zelikovsky, the approximation ratios of $\min\{2, \frac{3}{1+2\epsilon}\}$ and of $\min\{2, \frac{3}{3-2\sqrt{1-\bar{\epsilon}}}\}$, respectively.

On the other hand, we give new approximation lower bounds for the Minimum Edge Dominating Set problem in dense graphs. Assuming the Unique Game Conjecture, we show that it is NP-hard to approximate the Minimum Edge Dominating Set problem in everywhere ϵ-dense graphs with a ratio better than $\frac{2}{1+\epsilon}$ with $\epsilon \geq 1/2$ and $\frac{2}{2-\sqrt{1-\bar{\epsilon}}}$ with $\bar{\epsilon} \geq 3/4$ in average $\bar{\epsilon}$-dense graphs.

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1 Introduction

In this paper we consider the computational complexity of approximating the *Minimum Subset Edge Dominating Set* problem which generalizes the Minimum Edge Dominating Set problem. As a direct result, we obtain improved upper bounds for the Minimum Edge Dominating Set problem in everywhere and average dense graphs, i.e. graphs with bounded minimum and average vertex degree, respectively.

1.1 Problem Statement

An *edge dominating set* (for short EDS) of a finite undirected graph $G = (V, E)$ is a subset $M \subseteq E$ of edges such that each edge in E shares an endpoint with some edges in M. The *Minimum Edge Dominating Set* problem (for short MEDS problem) asks to find an edge dominating set of minimum cardinality $|M|$ (respectively minimum total weight in the weighted case).

For given graph $G = (V, E)$ the *Minimum Maximal Matching* problem (for short MMM problem) asks for a subset $M \subseteq E$ of non adjacent edges with minimal cardinality such that each edge in E shares an endpoint with some edge in M.

It has been noted long time ago that the Minimum Edge Dominating Set and the Minimum Maximal Matching problem admit optimal solutions of the same size and that an optimal solution of the MEDS problem can be transformed in polynomial time into an optimal solution of the MMM problem (cf. [YG80]), and vice versa.

The *Minimum Subset Edge Dominating Set* problem (for short MSED problem) is a generalization of the MEDS problem and is defined as follows: given a graph $G = (V, E)$ and a subset $S \subseteq V$, find a minimum cardinality EDS M of G with the property $S \subseteq \bigcup_{e \in M} e$.

For some $\epsilon, \bar{\epsilon} > 0$, we call a graph $G = (V, E)$ *everywhere ϵ-dense* if every vertex in G has at least $\epsilon|V|$ neighbors, and we call a graph $G = (V, E)$ *average $\bar{\epsilon}$-dense* if the average degree of the vertices in G is at least $\bar{\epsilon}|V|$, i.e. $\frac{\sum_{v \in V} deg(v)}{|V|} \geq \bar{\epsilon}|V|$.

1.2 Related Work

The MEDS problem is already referred to in Gary and Johnson [GJ79]. Even for planar or bipartite graphs of maximum degree 3 the MEDS problem remains NP-hard [YG80] in the exact setting. Some additional hard and polynomial time solvable classes of graphs were given by Horten and Kiliakos ([HK93]), and much more recently by Demange and Ekim ([DE08]). An inapproximability result was obtained by Chlebík and Chlebíková ([CC06]), who showed that it is NP-hard to approximate the MEDS problem within any factor better than $\frac{7}{6}$. They further showed that the MEDS problem is NP-hard to approximate within any constant less than $\frac{7+\epsilon}{6+2\epsilon}$, in graphs with minimum degree at least ϵn. A $2\frac{1}{10}$-approximation algorithm was given by Car et al. ([CFKP01]) for the *Minimum Weighted Edge Dominating Set* problem, a result which was improved to 2 by Fujito and Nagamochi ([FN02]).

Cardinal et al. achieved the first upper bound smaller than 2 for sufficiently dense graphs. More precisely, the obtained approximation ratio is asymptotic to $\min\{2, \frac{1}{\epsilon}\}$ in everywhere ϵ-dense graphs and to $\min\{2, \frac{1}{1-\sqrt{1-\bar{\epsilon}}}\}$ in average $\bar{\epsilon}$-dense graphs ([CLLLM05]). More recently Cardinal, Langerman, and Levy provided an improved bound on the approximation ratio for the MEDS problem in average dense graphs. This bound is asymptotic to $\frac{1}{1-\sqrt{(1-\epsilon)/2}}$, which is smaller than 2 when ϵ is greater than $\frac{1}{2}$ ([CLL09]).

1.3 Our Contributions

This work is the first best to our knowledge studying the approximation complexity of the MSED problem. We give an approximation algorithm that achieves the approximation ratio at most $\min\{2, \frac{3}{1+2|S|/|V|}\}$. For the special case of the MEDS problem in dense graphs, it yields by using the techniques of Karpinski and Zelikovsky for the dense Minimum Vertex Cover problem ([KZ97]) an approximation ratio of $\min\{2, \frac{3}{1+2\epsilon}\}$ for everywhere ϵ-dense graphs and $\min\{2, \frac{3}{3-2\sqrt{1-\bar{\epsilon}}}\}$ for average $\bar{\epsilon}$-dense graphs, respectively.

On the other hand, we construct an approximation preserving reduction from the Minimum Vertex Cover problem to the MEDS problem in dense graphs. Thus assuming the unique game conjecture (cf. [KR08]), it is NP-hard to approximate the MEDS problem in everywhere ϵ-dense graphs with a ratio better than $\frac{2}{1+\epsilon}$ with $\epsilon \geq \frac{1}{2}$ and $\frac{2}{2-\sqrt{1-\bar{\epsilon}}}$ with $\bar{\epsilon} \geq \frac{3}{4}$ in average $\bar{\epsilon}$-dense graphs. The same reduction shows that the MSED problem is UGC-hard to approximate within any constant better than $\frac{2}{1+\frac{|S|}{|V|}}$ with $2|S| \geq |V|$.

2 Subset Edge Dominating Set Problem

We start by introducing some basic notations and tools which are used in our algorithms. Afterwards we state our approximation algorithm for the MSED problem and prove the claimed result.

2.1 Definitions and Notations

Given a finite graph $G = (V, E)$ and a subset $S \subseteq V$, the induced subgraph $G[S]$ is defined as $(S, \{e \in E \mid e \subseteq S\})$. For a given set $M \subseteq E$ we introduce the notation $V(M) := \bigcup_{e \in M} e$.

The maximal matching heuristic is a standard algorithm that provides a 2-approximation for the Minimum Edge Dominating Set problem. It is perhaps one the simplest and best-known approximation algorithm. It consists in finding a collection of disjoint edges (a matching) that is maximal (with respect to edge inclusion) by iteratively removing adjacent vertices until no more edges are left in the graph.

In the Maximum Subset Matching problem (for short MSM problem), which generalizes the Maximum Matching problem, we are given a graph $G = (V, E)$ and $S \subset V$. The goal is to determine the maximum number of vertices of S that can be matched in a matching of G. Alon and Yuster considered this problem and introduced a randomized algorithm in [AY07]. The Maximum Subset Matching problem can be reduced to the Maximum Weighted Matching problem. Just assign to every vertex with both end-points in S weight 2, and edges from S to $V \backslash S$ weight 1. The currently fastest algorithm for maximum weighted matchings in general graphs is the algorithm of Gabow and Tarjan (see [GT91]).

In our setting, it runs in $\tilde{O}(\sqrt{|V|}(|E| + |S|^2)$ time. For a given graph $G = (V, E)$, $S \subseteq V$ and $U \subseteq V \backslash S$, let us denote by $MSM(G, S, U)$ the set of edges of a maximal subset matching in the graph $G[S \cup U]$ and S.

An important theorem of for many problems related to the Minimum Vertex Cover problem was proven by Nemhauser and Trotter (cf. [NT75]). It enables us to reduce the problem to instances in which the value of a minimum vertex cover is at least $\frac{1}{2}|V|$ together with other nice properties. Here we use a generalized version of the NT-Theorem given by Chlebík and Chlebíková.

Theorem. (Optimal Version of the NT-Theorem [CC04])
There exists a polynomial time algorithm that partitions the vertex set V of any graph G into three subsets $(V_0, V_1, V_{1/2})$ with no edges between V_0 and $V_{1/2}$ or within V_0 such that

1. *for any vertex cover VC of $G[V_{1/2}]$ it holds $|VC| \geq \frac{1}{2}|V_{1/2}|$*
2. *every minimum vertex cover C for G satisfies $V_1 \subseteq C \subseteq V_1 \cup V_{1/2}$ and $C \cap V_{1/2}$ is a minimum vertex cover for $G[V_{1/2}]$.*

Such a partition can be constructed by computing maximal matching of a specially constructed bipartite graph. The algorithm of Hopcroft and Karp is currently the fastest algorithm for maximum matching in bipartite graphs and runs in time $O(|E|\sqrt{|V|})$ (see [HK73]).

2.2 Algorithm $\mathcal{A}_{\mathbf{SEDS}}$

In order to explain the intuition behind the algorithm, notice that the set S needs to be covered with edges and we want to achieve it by a maximum matching which covers the whole set S. Clearly, we cannot expect that there always exists a perfect matching in $G[S]$. Instead we compute a maximal subset matching with endpoints in $V_1 \cup V_{1/2}$ for which we hope to have good vertex cover properties in $G[V \backslash S]$. The remaining vertices of S will be covered greedily. Finally, we take care of the remaining graph by applying the maximal matching heuristic.

We now present our main algorithm.

Algorithm $\mathcal{A}_{\mathbf{SEDS}}$

Input: Graph $G = (V, E)$, $S \subseteq V$

Set $M_1 := \emptyset$;
If $|S| > \frac{|V|}{4}$ **Then**
  Compute the NT-Partition $(V_0, V_1, V_{1/2})$ of $G[V \backslash S]$;
  If $|V_0| < 2|V_1|$ **Then**
    Compute $M_1 = MSM(G, S, V \backslash S)$;
  Else
    Compute $M_1 = MSM(G, S, V_1 \cup V_{1/2})$;
  EndIf
EndIf
Cover the remaining vertices of S greedily with edges M_r;
Compute the remaining graph $G' = G[V \backslash V(M_1 \cup M_r)]$;
Construct a maximal matching M_2 in G' by applying the maximum matching heuristic;

Output: $M_1 \cup M_r \cup M_2$

2.3 Analysis of $\mathcal{A}_{\mathbf{SEDS}}$

We now formulate our main theorem.

Theorem 2.1. *Given a graph $G = (V, E)$ and $S \subseteq V$, the algorithm $\mathcal{A}_{SEDS}$ has an approximation ratio at most* $min\left\{2, \frac{3}{1+2|S|/|V|}\right\}$.

Proof. Let OPT denote some optimal solution for the SMEDS problem and $EDS_{\mathcal{A}}$ the solution produced by algorithm $\mathcal{A}_{\text{SEDS}}$.
We have to consider three different cases:

Case $|S| \leq \frac{|V|}{4}$**:**
The algorithm covers the vertices of S greedily with edges, which means that we use at most $|S|$ edges. Since the maximal matching heuristic computes a solution as well for the MEDS problem as for the Minimum Vertex Cover problem (by choosing the endpoints of the constructed matching) with approximation ratio 2, our solution for the graph $G[V \backslash S]$ has at most as much edges as the cardinality of an optimal vertex cover VC_{OPT} of $G[V \backslash S]$. Consequently, the approximation ratio of the algorithm is bounded by

$$\frac{|EDS_{\mathcal{A}}|}{|OPT|} \leq \frac{|S| + 2|VC_{OPT}|}{\frac{1}{2}|S| + |VC_{OPT}|} \leq 2.$$

Case $|V_0| < 2|V_1|$**:**
First of all, the algorithm $\mathcal{A}_{\text{SEDS}}$ computes an maximum subset matching $M_1 := MSM(G, S, V \backslash S)$ of G and then covers the remaining vertices of S greedily with edges M_r (see figure 1).

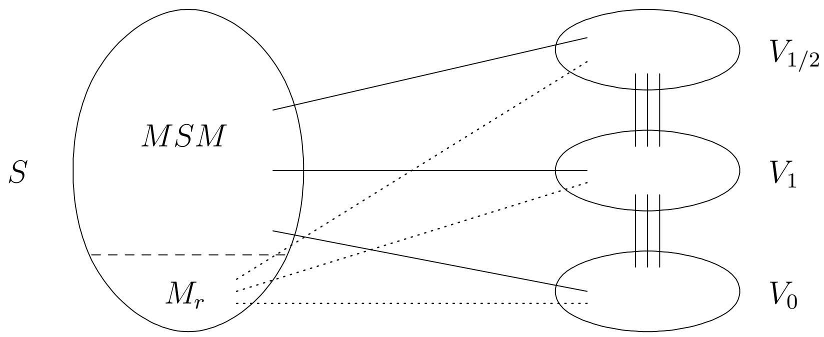

Figure 1: The Partition of G in the case of $|V_0| < 2|V1|$.

For the sake of the analysis, let us now consider a maximum subset matching $M^* := MSM(G^*, S, V(OPT) \backslash S)$ of the restricted graph $G^* = (V(OPT), OPT)$ and denote by M_R the edges contained in OPT to cover the vertices in $S \backslash V(M^*)$.

In the following we will bound the approximation ratio of the solution produced by algorithm $\mathcal{A}_{\text{SEDS}}$:

$$\begin{aligned}
\frac{|EDS_{\mathcal{A}}|}{|OPT|} &\leq \frac{\frac{1}{2}(|S|-|M_r|)+\frac{1}{2}|V_1|+\frac{1}{2}|V_{1/2}|+\frac{1}{2}|V_0|+|M_r|}{\frac{1}{2}(|S|-|M_R|)+\frac{1}{2}|V_1|+\frac{1}{4}|V_{1/2}|+|M_R|} && (1)\\
&\leq \frac{|S|+|V_1|+|V_{1/2}|+|V_0|+|M_r|}{|S|+|V_1|+\frac{1}{2}|V_{1/2}|+|M_R|} && (2)\\
&\leq \frac{|S|+|V_1|+|V_{1/2}|+|V_0|+|M_R|}{|S|+|V_1|+\frac{1}{2}|V_{1/2}|+|M_R|} && (3)\\
&\leq \frac{|S|+|V_1|+|V_{1/2}|+|V_0|}{|S|+|V_1|+\frac{1}{2}|V_{1/2}|} && (4)
\end{aligned}$$

In (1) we used the property of the NT-theorem $|V_1|+\frac{1}{2}|V_{1/2}|$ being a lower bound on the size of a vertex cover of the remaining graph $G[V\backslash S]$. Since OPT is contained in $E(G)$, it is clear that

$$|MSM(G^*, S, V(OPT)\backslash S)| \leq |MSM(G, S, V\backslash S)|$$

holds. Thus, we get $|M_r| \leq |M_R|$ which we used in (3).

To get (7) we are exploiting the property of the case $|V_0| < 2|V_1|$. The remaining part follows by simple algebraic calculation:

$$\begin{aligned}
\frac{|S|+|V_1|+|V_{1/2}|+|V_0|}{|S|+|V_1|+\frac{1}{2}|V_{1/2}|} &\leq \frac{3}{\frac{3|S|+3|V_1|+\frac{3}{2}|V_{1/2}|}{|S|+|V_1|+|V_{1/2}|+|V_0|}} && (5)\\
&\leq \frac{3}{\frac{|S|+3|V_1|+|V_{1/2}|}{|S|+|V_1|+|V_{1/2}|+|V_0|}+\frac{2|S|+\frac{1}{2}|V_{1/2}|}{|S|+|V_1|+|V_{1/2}|+|V_0|}} && (6)\\
&\leq \frac{3}{\frac{|S|+3|V_1|+|V_{1/2}|}{|S|+3|V_1|+|V_{1/2}|}+\frac{2|S|+\frac{1}{2}|V_{1/2}|}{|V|}} && (7)\\
&\leq \frac{3}{1+\frac{2|S|+\frac{1}{2}|V_{1/2}|}{|V|}} \leq \frac{3}{1+2\frac{|S|}{|V|}} && (8)
\end{aligned}$$

Case $|V_0| \geq 2|V_1|$:
Unlike to the previous case the algorithm $\mathcal{A}_{\text{SEDS}}$ computes a maximum subset matching $MSM(G, S, V_1 \cup V_{1/2})$ of G (see figure 2). As before M_r and M_R are the sets of edges to cover the remaining vertices of S, where $V(M_R) \cap S$ are the vertices left uncovered by a maximum subset matching $MSM(G^*, S, (V_1 \cup V_{1/2}) \cap V(OPT))$ of $G^* := (V(OPT), OPT)$.

Now, let us bound the approximation ratio of the solution produced by the algorithm:

$$\frac{|EDS_{\mathcal{A}}|}{|OPT|} \leq \frac{\frac{1}{2}(|S|-|M_r|)+|V_1|+\frac{1}{2}|V_{1/2}|+|M_r|}{\frac{1}{2}(|S|-|M_R|)+\frac{1}{4}|V_{1/2}|+|M_R|} \tag{9}$$

$$\leq \frac{|S|+2|V_1|+|V_{1/2}|+|M_r|}{|S|+\frac{1}{2}|V_{1/2}|+|M_R|} \tag{10}$$

$$\leq \frac{|S|+2|V_1|+|V_{1/2}|+|M_R|}{|S|+\frac{1}{2}|V_{1/2}|+|M_R|} \tag{11}$$

$$\leq \frac{|S|+2|V_1|+|V_{1/2}|+|V_1|}{|S|+\frac{1}{2}|V_{1/2}|+|V_1|} \tag{12}$$

We obtain (9) since the maximal matching heuristic computes an EDS of $G[V\backslash(S \cup V(M_2 \cup M_r))]$ with approximation ratio 2. Thus, the cardinality of the produced solution is bounded by $|V_1| + \frac{1}{2}|V_{1/2}|$. In (11) we use the maximality of the constructed maximum subset matching. In particular, it holds $|M_r| \leq |M_R|$.

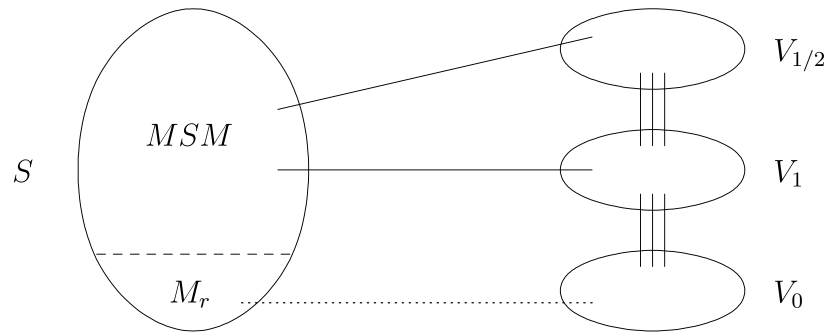

Figure 2: The Partition of G in the case of $|V_0| \geq 2|V1|$.

Since the edges M_r were chosen greedily to cover the remaining vertices in S, we cannot ensure that the endpoints of M_r have good vertex cover properties in $G[V\backslash S]$. In contrast, some of the vertices of $V(M_R) \cap V_0$ could be used to cover edges between V_0 and V_1. Nevertheless, the number of such edges is bounded by $|V_1|$, since $|V_1|+\frac{1}{2}|V_{1/2}|$ is a lower bound on the cardinality of an optimal vertex cover of $G[V\backslash S]$. In this way, we use $|M_R| \geq |V_1|$ to attain (12).

To get (15) we are exploiting the property of the case $|V_0| \geq 2|V_1|$. The rest of the proof follows by simple algebraic calculation:

$$\frac{|S| + 3|V_1| + |V_{1/2}|}{|S| + |V_1| + \frac{1}{2}|V_{1/2}|} \leq \frac{3}{\frac{3|S|+3|V_1|+\frac{3}{2}|V_{1/2}|}{|S|+3|V_1|+|V_{1/2}|}} \quad (13)$$

$$\leq \frac{3}{\frac{|S|+3|V_1|+|V_{1/2}|}{|S|+3|V_1|+|V_{1/2}|} + \frac{2|S|+\frac{1}{2}|V_{1/2}|}{|S|+3|V_1|+|V_{1/2}|}} \quad (14)$$

$$\leq \frac{3}{1 + \frac{2|S|+\frac{1}{2}|V_{1/2}|}{|V|}} \leq \frac{3}{1 + 2\frac{|S|}{|V|}} \quad (15)$$

□

3 MEDS Problem in Dense Graphs

In this section we consider the Minimum Edge Dominating Set problem in dense graphs. Firstly, we start with a observation of fundamental importance to our analysis.

Observation 3.1. *Given a connected graph $G = (V, E)$ and an optimal EDS M of G. There is a vertex $v \in V$ with $N(v) \subseteq V(M)$.*

Proof. If M covers the whole vertex set V, then we have nothing to show. Otherwise the whole neighborhood of a vertex $v \in V \backslash V(M)$ belongs to $V(M)$ to cover the edges incident to v. □

This observation gives us a simple proof of the analysis of the approximation ratio of the maximal matching heuristic in dense graphs studied by Cardinal et al. (see [CLLLM05]). Since the cardinality of an optimal EDS of an everywhere ϵ-dense graph $G = (V, E)$ can be lower bounded by $\min_{v \in V}\{|N(v)|\}/2 \geq \epsilon|V|/2$ and the worst case solution of the maximal matching heuristic is a maximum matching, the approximation ratio is bounded by $\min\{2, \frac{|V|/2}{\epsilon|V|/2}\}$.

Next, we want to derive an equivalent statement for average $\bar{\epsilon}$-dense graphs. We need a lemma which was proven by Karpinski and Zelikovsky.

Lemma 3.1. *[KZ97] Given an $\bar{\epsilon}$ average dense graph $G = (V, E)$ and let W be the set of $(1 - \sqrt{1 - \bar{\epsilon}})|V|$ vertices with highest degree. Then every vertex of W has degree at least $|W|$.*

As a direct consequence, we get the following

Corollary 3.1. *Given an $\bar{\epsilon}$-average dense Graph $G = (V, E)$. The cardinality of an optimal EDS M is at least $(1 - \sqrt{1 - \bar{\epsilon}})|V|/2$.*

Proof. If the whole set W of $(1-\sqrt{1-\bar{\epsilon}})|V|$ vertices with highest degree belongs to $V(M)$, we have nothing to show. Otherwise the neighborhood of a vertex $v \in W \backslash V(M)$ is a subset of $V(M)$. According to lemma 3.1 the degree of this vertex v is at least $(1-\sqrt{1-\bar{\epsilon}})|V|$. Therefore, the cardinality of M can be lower bounded by $|N(v)|/2 \geq (1-\sqrt{1-\bar{\epsilon}})|V|/2$. □

Analogously, one can easily deduce similarly to observation 3.1 that the maximal matching heuristic computes an EDS in average $\bar{\epsilon}$-dense graphs with approximation ratio at most $\min\{2, (1-\sqrt{1-\bar{\epsilon}})^{-1}\}$ as analyzed in [CLLLM05].

We are ready to state the algorithm for the dense MEDS problem:

Algorithm $\mathcal{A}_{\mathbf{DEDS}}$

Input: Graph $G = (V, E)$

ForAll $v \in V$
    compute $\mathcal{A}_{\text{SEDS}}(G, N(v))$;
EndForAll
Let M_1 be the solution with smallest cardinality among $\{\mathcal{A}_{\text{SEDS}}(G, N(v)) \mid v \in V\}$;
Let W be the set of $(1-\sqrt{1-\bar{\epsilon}})|V|$ vertices with highest degree;
Compute $M_2 := \mathcal{A}_{\text{SEDS}}(G, W)$;
ForAll $v \in W$
    compute $\mathcal{A}_{\text{SEDS}}(G, N(v))$;
EndForAll
Let M_3 be the solution with smallest cardinality among $\{\mathcal{A}_{\text{SEDS}}(G, N(v)) \mid v \in W\}$;

Output: The best solution among M_1, M_2 and M_3

Corollary 3.2. *The algorithm DEDS has an approximation ratio at most* $\frac{3}{1+2\epsilon}$ *for* ϵ*-everywhere dense graphs and at most* $\frac{3}{3-2\sqrt{1-\bar{\epsilon}}}$ *for* $\bar{\epsilon}$*-average dense graphs.*

Proof. Given an ϵ-everywhere dense graph $G = (V, E)$ and an optimal EDS M, $V(M)$ contains always the neighborhood $N(v)$ of a vertex $v \in V$ because of observation 3.1. By exhaustive search we find the right vertex v and use the algorithm for the MSED problem. We get a solution with an approximation ratio at most $\frac{3}{1+2\frac{|N(v)|}{|V|}} \leq \frac{3}{1+2\frac{\epsilon|V|}{|V|}}$.

In the case of $\bar{\epsilon}$-average dense graphs we have to consider two cases. If there is a vertex $v \in W$, which does not belong to $V(M)$, then we use the

same argumentation as before. Since the smallest degree of a vertex in W is at least $(1-\sqrt{1-\bar{\epsilon}})|V|$, the approximation ratio can be bounded as follows: $\frac{3}{1+2\frac{|N(v)|}{|V|}} \leq \frac{3}{1+2(1-\sqrt{1-\bar{\epsilon}})} = \frac{3}{3-2\sqrt{1-\bar{\epsilon}}}$.
Otherwise the whole set W belongs to $V(M)$. Since the cardinality of W is $(1-\sqrt{1-\bar{\epsilon}})|V|$, the corollary follows from theorem 2.1. □

4 Approximation Hardness Results

Assuming the Unique Game Conjecture (see [K02]), we provide new lower bounds on efficient approximability for everywhere ϵ-dense (resp. average $\bar{\epsilon}$-dense) instances of the MEDS problem with $1/2 \leq \epsilon$ (resp. with $3/4 \leq \bar{\epsilon}$).

Using the hardness result of Khot and Regev [KR08] as a starting point, we construct an approximation preserving reduction from the Minimum Vertex Cover problem to dense instances of the MEDS problem.

We now formulate our result in the form of a theorem.

Theorem 4.1. *For every $\delta > 0$ it is unique game conjecture hard to approximate the everywhere ϵ-dense MEDS problem with $\epsilon \geq 1/2$ (resp. average $\bar{\epsilon}$-dense MEDS problem with $\bar{\epsilon} \geq 3/4$) to within $\frac{2}{1+\epsilon} - \delta$ (resp. $\frac{2}{2-\sqrt{1-\bar{\epsilon}}} - \delta$).*

Proof. Given a general instance $G = (V, E)$ of the Minimum Vertex Cover problem with $n := |V|$, we construct an everywhere ϵ-dense graph $G' = (V', E')$ as an instance of the dense MEDS problem.

In order to construct G' we add a clique K of size $\frac{\epsilon}{1-\epsilon}n$ to G and connect each vertex of V with every vertex of K (Figure 3 illustrates this construction). Hence, every vertex of G' has a vertex degree at least

$$\frac{\epsilon}{1-\epsilon} \cdot n = \frac{\epsilon}{1-\epsilon} \cdot \frac{n'}{1+\frac{\epsilon}{1-\epsilon}} = \frac{\epsilon}{1-\epsilon} \cdot \frac{n'}{\frac{1-\epsilon}{1-\epsilon}+\frac{\epsilon}{1-\epsilon}} = \epsilon \cdot n'.$$

Every feasible solution of the Minimum Edge Dominating Set problem EDS of G' can be transformed in polynomial time into a vertex cover VC of the original graph G: In order to simplify the analysis we define an operation called *switch*. Let C be the simple cycle of length 4. A *switch* replaces two arbitrary non-adjacent edges of C with the remaining ones. We can restrict ourselves to edge dominating sets of G' with the property $E \cap EDS = \emptyset$, since for every edge $\{u, w\} \in E \cap EDS$ there exist the edges $\{x, y\} \in EDS \cap K$,$\{u, x\}$ and $\{y, w\}$ to form a cycle of length 4 on which we can perform a *switch*. Now it is easy to see that the set $V(EDS) \cap V(G)$ is a vertex cover of G. Moreover, an optimal edge dominating set OPT_{EDS} of G' can be transformed in this way into an optimal vertex cover OPT_{VC} of

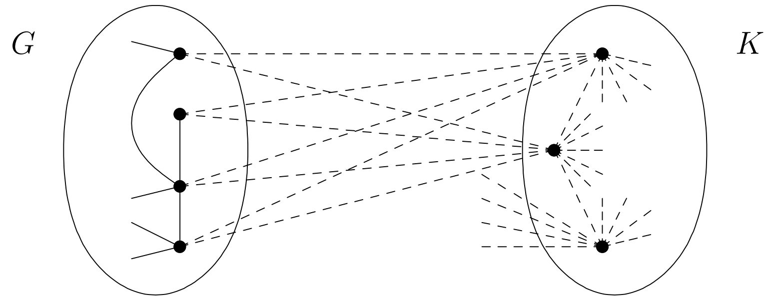

Figure 3: Construction of G'.

G in polynomial time. Since the whole vertex set of K should be contained in $V(EDS)$, we get $2|EDS| - \frac{n\epsilon}{1-\epsilon}$ as an upper bound on the cardinality of the derived vertex cover VC. Assuming the existence of an approximation algorithm with approximation ratio $\left(\frac{2}{1+\epsilon} - \delta\right)$ with $\delta > 0$ for the everywhere ϵ-dense Minimum Edge Dominating Set problem, we would achieve an approximation algorithm for the Minimum Vertex Cover problem with the following approximation ratio:

$$
\begin{aligned}
|VC| &\leq 2 \cdot |EDS| - \frac{n\epsilon}{1-\epsilon} \\
&\leq \left(\frac{2}{1+\epsilon} - \delta\right) \cdot 2 \cdot |OPT_{EDS}| - \frac{n\epsilon}{1-\epsilon} \\
&\leq \left(\frac{2}{1+\epsilon} - \delta\right) \cdot 2 \cdot \left(\frac{|OPT_{VC}|}{2} + \frac{n\epsilon}{2(1-\epsilon)}\right) - \frac{n\epsilon}{1-\epsilon} \\
&\leq \frac{2}{1+\epsilon} \cdot |OPT_{VC}| + \frac{2n\epsilon}{(1-\epsilon)(1+\epsilon)} - \frac{n\epsilon(1+\epsilon)}{(1-\epsilon)(1+\epsilon)} - \delta \cdot |OPT_{VC}| \\
&\leq \frac{2}{1+\epsilon} \cdot |OPT_{VC}| + \frac{n\epsilon(1-\epsilon)}{(1-\epsilon)(1+\epsilon)} - \delta \cdot |OPT_{VC}|
\end{aligned}
$$

Using the NT-theorem, we can restrict ourselves to optimal solutions for the Minimum Vertex Cover problem with sizes at least $n/2$.

$$
\begin{aligned}
|VC| &\leq \frac{2}{1+\epsilon} \cdot |OPT_{VC}| + \frac{2\epsilon}{1+\epsilon} \cdot |OPT_{VC}| - \delta \cdot |OPT_{VC}| \\
&\leq (2-\delta) \cdot |OPT_{VC}|
\end{aligned}
$$

This is a contradiction to the vertex cover hardness result by Khot and Regev [KR08] based on the Unique Game Conjecture.

In the case of average $\bar{\epsilon}$-dense instances of the Minimum Edge Dominating Set problem, we can use the same construction yielding a hardness result of $\frac{2}{2-\sqrt{1-\bar{\epsilon}}}$ with $\bar{\epsilon} \geq 3/4$. □

Using the same reduction for the MSED problem with $S = V(K)$, we get the following

Corollary 4.1. *For every $\delta > 0$ and $2|S| \geq |V|$, it is UGC-hard to approximate the MSED problem within $\frac{2}{1+\frac{|S|}{|V|}} - \delta$.*

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