Polynomial Time Approximation of Dense Weighted Instances of MAX-CUT (Revised Version)*

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Abstract

We give the first polynomial time approximability characterization of dense weighted instances of MAX-CUT, and some other dense weighted $\mathcal{NP}$-hard problems in terms of their empirical weight distributions. This gives also the first almost sharp characterization of inapproximability of unweighted 0,1 MAX-BISECTION instances in terms of their density parameter.

Key words: Randomized Algorithms, Approximation Schemes, MAX-CUT, MAX-BISECTION, Approximation Hardness, Density Classes

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1 Introduction

Significant results concerning polynomial time approximation schemes (PTASs) for "dense" instances of several $\mathcal{NP}$-hard problems such as MAX-CUT, MAX-k-SAT, BISECTION, DENSE-k-SUBGRAPH, and others have been obtained recently in Arora, Karger and Karpinski [AKK95], Fernandez de la Vega [FV96], Arora, Frieze and Kaplan [AFK96], Frieze and Kannan [AK97]. Still more recently, the approximability of dense instances of $\mathcal{NP}$-hard problems has been investigated from the point of view of the query complexity. Goldreich, Goldwasser and Ron [GGR96] show that a constant size sample is sufficient to test whether a graph has a cut of a certain size. Frieze and Kannan [AK97], obtain quick approximations for all dense MAX-SNP problems. Recall that a PTAS for a given optimization problem is a family (A_ϵ) of algorithms indexed by a parameter $\epsilon \in (0, \infty)$ where each algorithm runs in polynomial time and, for each ϵ, the algorithm A_ϵ has approximation ratio $1 - \epsilon$ (or $1 + \epsilon$ for a minimization problem). In most cases, the instances are graphs, and a dense graph is defined as a graph with $\Theta(n^2)$ edges where n is the number of vertices. (In some cases, the algorithms apply only to graphs with minimum degree $\Theta(n)$.) Some of the problems considered in the papers mentioned above, such as MAX-CUT, are MAX-SNP-hard, and thus, if $\mathcal{P} \neq \mathcal{NP}$, have no PTASs when the set of instances is not restricted. Let us also mention that the PTASs in [FV96], [AK97] and [GGR96] are *efficient* in the sense of Cesati and Trevisan [CS97].

The natural instances of optimization problems (see, e.g., [GJ79]) involve weights while the results mentioned above deal mainly with the 0,1 case. The purpose of this paper is to examine how these results can be extended to the weighted case. We want to define a concept of density for the weighted case which ensures that our algorithms, possibly with minor modifications, work in the corresponding *dense* classes of instances and such that the *non-dense* classes are not approximable under a standard intractability assumption. For the sake of simplicity, we concentrate here on MAX-CUT. In fact, for technical reasons, we start by considering MAX-BISECTION, which is MAX-CUT restricted to cuts with equal sides. (MAX-BISECTION is also called MAX-50/50-CUT or MAX-EQUI-CUT.) Our results extend easily to other MAX-SNP-hard problems such as MAX-2SAT or MAXIMUM ACYCLIC SUBGRAPH. We remark in passing that the methods of [AKK95] and [FV96] give a PTAS for MAX-BISECTION.

We note that weight problems have been briefly considered in [GGR96] and [AK97]. In both papers, the authors evaluate the increase of the computation time of their algorithms when one allows weights belonging to some fixed interval $[0, a]$ instead of 0,1 weights. Weight problems are also considered in a recent paper [TR97].

2 Overview

We define first in sections 2 and 3 our dense classes of weighted instances via classes of distribution functions (d.f.'s for short) of the weights. They clearly grasp the intuitive, and standard notions of dense instances of combinatorial optimization. We prove in section 5 that MAX-CUT has a PTAS in any dense class of weighted instances according to our definition.

Then, we should ideally prove that both MAX-BISECTION and MAX-CUT are MAX-SNP hard on any fixed set of weighted instances which is not dense. Please note that we aim at characterizing the inherent inapproximability of optimization problems in their density parameter only, and in the 0,1 case, the following issue is not clear on how fast the densities of our instances should tend to 0:

Hardness of MAX-CUT on a non-dense set of 0,1 instances: Let $(d_n)_{n=0,1\dots}$ be a sequence of rational numbers tending to 0 as $n \to \infty$. Is it always true that MAX-CUT is Max-SNP-hard when restricted to the set of all graphs whose densities belong to (d_n)?

The answer is of course yes, if we replace the sequence (d_n) by any interval $[d_o, 0[$. It can be negative if the denominators of the d_n's are huge (see section 7) and, to our annoyance, we could only find a rather lengthy proof that the answer is yes subject to a further condition on the rate of decrease of the d_n's (see section 7, Theorem 6).

The rest of section 6 is devoted to the proof of the MAX-SNP hardness of MAX-BISECTION and MAX-CUT on non-dense sets of weighted instances, using reductions to non-dense 0,1 instances. The last section contains a summary and open problems.

3 Dense Families of Instances

3.1 Definition of a Dense family

In as much as density requirements come in, any given instance is a set of non-negative real numbers (the weights) or rather a multiset. Let us associate to this instance the empirical distribution function of the weights:

$$F(x) = \frac{2}{n(n-1)} \sum_{x_i \leq x} m_i, \quad x \in \mathbb{R}^+$$

where m_i denotes the multiplicity of the weight x_i in the instance and n is the number of vertices.

We define our density classes in terms of families of weight distribution functions. More precisely:

(i) To each d.f. F with support in $\mathbb{R}^+$, we associate the set I_F of all weighted graphs whose empirical weight distribution coincides with F.

(ii) To each set $\mathcal{F}$ of d.f.'s we associate the set of instances

$$I_{\mathcal{F}} = \cup_{F \in \mathcal{F}} \ I_F$$

Thus, we shall define below *dense* sets of d.f.'s having in mind the sets of instances to which they correspond according to (i) and (ii).

Stated in different words, our setting of density classes in terms of weight d.f.'s, amounts exactly to saying that, with any fixed instance belonging to a density class, we also include in this class, all the instances which have the same weight distribution. This assumption is of course very natural.

Clearly, our d.f.'s need to have finite discrete support and rational individual probabilities. (We don't dwell here about the nature of the values in the support. For definitness, let us say that they are also rational.) We call such d.f.'s *representable*. Conversely, the set of instances corresponding to a representable d.f. F with individual probabilities having smallest common denominator D, say, is given by

$$I_F = \cup_{\{n:2D|n(n-1)\}} \mathcal{G}_n$$

where $\mathcal{G}_n$ is the set of weighted graphs on n vertices whose empirical weight distribution coincides with F. Notice that I_F is infinite for any representable F. For convenience, when representativity is not essential, we state on various occasions our theorems in terms of arbitrary d.f.'s. (not necessarily having finite or even discrete ranges).

We can assume that the mean of the weights in each instance is equal to 1, since, when we divide all the weights by their mean, say m, we also divide the values of the objective function by m so that the approximation ratios are unaffected. (We assume that the weights are not all 0.) We shall say that a familly with all expectations equal to 1 is *standardized.* We consider mostly, but not always standardized families. (Here and all along the paper, we speak with some abuse of language, of the expectation of a d.f. F meaning the expectation of a random variable with d.f. F.)

We can now state our definition of a dense family of d.f.'s.

Definition 1 (Dense families of standardized d.f.'s) *Let $\mathcal{F} = (F_j)_{j \in \mathcal{J}}$ family of integrable d.f.'s with supports contained in $\mathbb{R}^+$ and all expectations equal to 1. For each $j \in \mathcal{J}$ and each $k \in \mathbb{N}$, define*

$$M_{j,k} = \frac{1}{k} \sum_{i=1}^{k} X_{j,i}$$

where the $X_{j,i}$ are independent r.v.'s each with d.f. F_j.

We say that the family $\mathcal{F}$ is dense if and only if, for each $j \in \mathcal{J}$, the sequence $(M_{j,k})_{k=1,2,..}$ converges in probability to 1, and moreover, this convergence is uniform for $j \in \mathcal{J}$.

In other words, $\mathcal{F}$ is dense iff there exists a function $n_\epsilon = n(\epsilon) : (0, 1] \to \mathbb{N}$ such that the inequalities

$$\mathsf{Pr}\ [|M_{j,k} - 1| \leq \epsilon] \geq (1 - \epsilon),\ \ k \geq n_\epsilon \tag{1}$$

hold for each ϵ and simultaneously for all j, with an n_ϵ which depends only on ϵ (and not on j).

Definition 2 *A family $\mathcal{F}$ of standardized d.f.'s which is not dense is called a non-dense family*

Definitions 1 and 2 are extended in the obvious way to non-standardized families: a family of distributions is dense (resp. non-dense) iff the corresponding standardized family is dense (resp. non-dense).

In the next section we identify some natural dense families of d.f.'s.

3.2 Some Dense Families

Recall the law of large numbers: If X has a finite mean EX, then the means of the partial sums of a sequence of independent random variables each distributed as X converges in probability to EX. This implies immediately the next proposition.

Proposition 1 *Any finite set of standardized integrable d.f.'s) with support in $\mathbb{R}^+$ is a dense family*

The following assertion can easily be checked.

Proposition 2 *The family of all integrable d.f.'s is not dense*

In the 0,1 case, which plays a key role in our proofs, we can represent a family $\mathcal{F} = \{F_i : i = 0, 1, ...\}$ by the set, say $\mathcal{D} = \{d_o, d_1, ...\}$ of the densities (in the standard sense of this word) of the corresponding instances. (Notice that, because of our scaling, F_i puts probability $1 - d_i$

on 0 and probability d_i on the point d_i^{-1}.) It is easy to see that $\mathcal{F}$ is dense in the sense of definition 1 iff $\mathcal{D}$ is bounded away from 0. (The converse statement posed serious problems to us, see section 6.1.) Thus, the corresponding set of instances is also dense in the standard sense. Note that this is the same as saying that the variances of the F_i''s are bounded from above.. This leads to the following more general class of dense families.

Proposition 3 *For each $s \geq 0$ the family*

$$\mathcal{F}_s = \left\{ F_X : \frac{\mathrm{Var} X}{(EX)^2} \leq s \right\}$$

is dense.

PROOF The proof is straightforward by using Chebyshev's inequality. □

The last example can be generalized as follows.

Proposition 4 *For each pair (r, C) where $r \in (1, +\infty)$ and $C \in \mathbb{R}^+$, the family of standardized d.f.'s $\{F_X\}$ satisfying each*

$$\frac{1}{(EX)^r} \int_0^\infty x^r dF_X(x) \leq C \tag{2}$$

is dense.

REMARK. Since our r.v.'s are generally discrete, integrals of the form

$$\int f(x) dF(x)$$

are interpreted as Riemann-Stielges integrals.

PROOF Fix $r \in (1, +\infty]$ and $C \in \mathbb{R}^+$ and let $\mathcal{F}$ be the corresponding family of d.f.'s defined in proposition 4 where we can suppose $EX = 1$ for every X. The inequality (2) gives immediately, for any $t \in \mathbb{R}^+$,

$$1 - F(t) = \int_t^\infty dF(x) \leq Ct^{-r},$$

We have thus

$$t(1 - F(t)) \leq Ct^{1-r}$$

whose right hand side tends to 0 when $t \to \infty$, uniformly for $F \in \mathcal{F}$. Anticipating our characterization of the dense families (see Theorem 1 in the next section), we deduce that $\mathcal{F}$ is dense. □

3.3 Some Non-Dense Families of d.f.'s

We present in this section some examples of non-dense families of d.f.'s. We begin with the 0,1 case which plays a key role in our proofs. Assume now that $\mathcal{F}$ is not dense, i.e. the set $\mathcal{D}$ is not bounded away from 0. Then, since MAX-CUT is MAX-SNP-hard and is approximable on dense sets, one would anticipate that MAX-CUT is also MAX-SNP-hard when restricted to the set of instances corresponding to $\mathcal{D}$. (The MAX-SNP-hardness of MAX-CUT tell us only that MAX-CUT is MAX-SNP-hard for any set of densities $\mathcal{D}$ containing an interval

$(0,\alpha]$, $\alpha>0$.) We only have a partial answer to this question (see theorem 6). For instance, we do not know the answer for the family $\mathcal{F}$ which corresponds to the set of densities

$$\mathcal{D}=\left\{2^{-2^{2^i}}:i=0,1,...\right\}$$

Let us give an example of a non-dense family $\mathcal{F}$ corresponding to weighted instances. It may not seem very natural but it examplifies the role of the tail in the non-density condition.

Proposition 5 *Let X and Y be non-negative and have distributions F_X and F_Y and means $EX<1$ and $EY>1$. For each $k\geq 1$ let Y_k have the d.f. F_{Y_k} defined by $F_{Y_k}(t)=F_Y(k^{-1}t)$, $t\in\mathbb{R}^+$. (Thus Y_k is distributed as Y with the scaling factor k.) Put*

$$\alpha_k=\frac{kEY-1}{kEY-EX},\qquad \beta_k=1-\alpha_k=\frac{1-EX}{kEY-EX}$$

Define for k=1,..., F_k by $F_k(t)=\alpha_kF_X(t)+\beta_kF_{Y_k}(t)$, $t\in\mathbb{R}^+$. Then, the family $\mathcal{F}=\{F_k:k=1,..\}$ is not dense

PROOF Note first that the expectations of the F_k are all equal to one. Anticipating again the characterization of the non-dense families given in the next section just after theorem 1, we have to find an $\eta>0$ such that, for any arbitrary large $x\in\mathbb{R}^+$, there is a k such that $x(1-F_k(x))\geq\eta$.
By the definition of F_k, we have

$$1-F_k(x)\geq\beta_k(1-F_Y(k^{-1}x))=\frac{(1-F_Y(k^{-1}x))(1-EX)}{kEY-EX}.$$

Fix any $z>0$ with $F_Y(z)<1$, say $F_Y(z)=1-a$. Then, for $x=kz$, we have

$$x(1-F_k(x))=kz(1-F_Y(z))\geq\frac{kz(1-EX)}{kEY-EX}$$

The last expression is asymptotic to $\frac{z(1-EX)}{EY}$. This concludes the proof with $\eta=\frac{z(1-EX)}{2EY}$, say. □

□

4 Characterization of the Dense Families

The following theorem characterizes the dense families. Once again, this theorem, alike Definition 1, is stated in terms of arbitrary (not necessarily representable) d.f.'s.

Theorem 1 *Let $\mathcal{F}=(F_j)_{j\in\mathcal{J}}$ be a family of non-negative integrable d.f.'s and assume all expectations equal to 1.*

The family $\mathcal{F}$ is dense in the sense of Definition 1 if and only if one of the following conditions (i) and (ii) holds:

(i) For each j and each $x\in\mathbb{R}^+$, define $\tau_j(x)=x(1-F_j(x))$. There is a function $\tau_o(x)$ tending to 0 as $x\to\infty$ and such that the inequalities

$$\tau_j(x)\leq\tau_o(x) \tag{3}$$

hold for each pair (j, x).

(ii) For each j and each $x \in \mathbb{R}^+$, define

$$s_j(x) = \int_x^\infty y dF_j(y). \tag{4}$$

There is a function $s_o(x)$ tending to 0 as $x \to \infty$ and such that the inequalities

$$s_j(x) \leq s_o(x) \tag{5}$$

hold for each pair (j, x).

Stated in other words, letting X_j denote a random variable with d.f. F_j, condition (ii) says that the X_j are uniformly integrable.

We shall also use occasionally the following characterization of the *non-dense* families.

Assume that $\mathcal{F}$ is not dense and that all expectations are equal to 1. Then there is an $\eta > 0$ such that, for any arbitrary large $y \in \mathbb{R}^+$, there is an $F \in \mathcal{F}$ with

$$y(1 - F(y)) \geq \eta. \tag{6}$$

To see this, note that the contrary would state:

$\forall \eta > 0 \ \exists y(\eta) \in \mathbb{R}^+$ s.t. $y(\eta)(1 - F_j(y(\eta))) < \eta$

for every $F_j \in \mathcal{F}$

Then, putting $y_k = y(2^{-k})$, we could define an τ_o for $\mathcal{F}$ by $\tau_o(x) = 2^{-k}$ for $y_k \leq x < y_{k+1}$, which contradicts the assumption that $\mathcal{F}$ is not dense.

PROOF (of Theorem 1). Let us see first that ii) implies i). Indeed, assume that ii) holds with some function $s_o(.)$. Now we have clearly $\tau_j(x) \leq s_j(x)$ for all j and x. Thus, condition i) holds by choosing $\tau_o(x) \equiv s_o(x)$. Now it suffices to show that (ii) is necessary and (i) sufficient.

The fact that condition (i) implies that the family $\mathcal{F}$ is dense in the sense of Definition 1 can be established easily by adapting the proof of the law of large numbers in order to get an effective bound on the sample size. Actually, we will adapt a proof of Feller (see [Fe]) that he uses to show the convergence of the means of sums of independent r.v.'s to a not necessary constant specified function. The speed of convergence is governed by the function τ. Let us write

$$S_n = X_1 + ... + X_n$$

where the X_i are independent with the common d.f. F with expectation 1. Let us define new r.v.'s X_i' by truncation at level n:

$$X_i' = X_i \text{ if } X_i \leq n, \qquad X_i' = 0 \text{ if } X_i > n.$$

Put

$$S_n' = X_1' + ... + X_n', \qquad m_n' = E(S_n') = nE(X_1').$$

Then,

$$P[|S_n - m_n'| > t] \leq P[|S_n' - m_n'| > t] + P[S_n \neq S_n'].$$

Putting $t = n\epsilon$ and applying Chebyshev's inequality to the first term on the right, we get

$$\textsf{Pr}\ [|S_n' - m_n'| > t] \leq \frac{1}{n\epsilon^2} E(X_1'^2) + nP[X_1 > n] \tag{7}$$

Put

$$\sigma(t) = \int_0^t x^2 dF(x).$$

Then, an integration by parts gives

$$\begin{aligned} \sigma(n) &= -n\tau(n) + 2\int_0^x \tau(x)dx \\ &\leq 2\int_0^x \tau(x)dx. \end{aligned}$$

(Recall that $\tau(x) = x(1 - F(x))$.) We have thus, for each n,

$$\textsf{Pr}\ \left[\left|\frac{S_n}{n} - EX_1'\right| \geq \epsilon\right] \leq \frac{2}{n\epsilon^2}\int_0^x \tau(x)dx + \tau(n)$$

Since EX_1' tends to $EX_1 = 1$ uniformly for $F \in \mathcal{F}$ as $n \to \infty$, this implies

$$\textsf{Pr}\ \left[\left|\frac{S_n}{n} - 1\right| \geq 2\epsilon\right] \leq \frac{2\int_0^x \tau(x)dx}{n\epsilon^2} + \tau(n) \tag{8}$$

for sufficiently large n. In order to prove that the right side tends to 0 again uniformly whenever $\tau(t) \leq \tau_o(t)$ with a $\tau_o(t) \to 0$, it suffices to prove that we have then

$$\int_0^n \tau(x)dx = o(n).$$

For this, choose an arbitrary small η and put $t = \tau_o^{-1}(\eta)$ Then,

$$\int_0^n \tau(x)dx \leq \int_0^t \tau(x)dx + \eta(n-t) \leq 2\eta n$$

for sufficiently large n and we are done. This concludes the proof of the sufficiency of condition (i).

For the only if direction, suppose that $\mathcal{F}$ does not satisfy to condition 5. Thus there exists, for an arbitrarily large y, an $F \in \mathcal{F}$ with

$$\int_y^\infty x dF(x) = \eta_y, \tag{9}$$

say, where $\eta \leq \eta_y \leq 1$. This implies of course $F(y) \geq 1 - \eta_y y^{-1}$. We claim that sample size $n = y\eta_y^{-1}$ does not suffice in order to estimate the expectation. This will conclude the proof since y is arbitrarily large.

Let us thus fix $n = y\eta_y^{-1}$. With probability

$$F(y)^n = (1 - \frac{\eta_y}{y})^{\frac{y}{\eta_y}} \geq e^{-1.1}, \tag{10}$$

(for sufficiently large y), all the points in the sample lie on the left-side of y. Now let Z be distributed as X conditioned by $X \leq y$. Then,

$$EZ = \frac{1-\eta}{F(y)} \leq 1 - \frac{9\eta}{10}, \tag{11}$$

for sufficiently large n. Let M denote the mean of the sample and let M_c denote the mean of n independent r.v.'s each distributed as Z. Set $p = \mathrm{P}[M_c \leq 1 - \eta/10]$. The inequality (11) implies clearly

$$1 - \frac{9\eta}{10} \geq (1-p)(1 - \frac{\eta}{10}).$$

This implies $p \geq \frac{4\eta}{5}$. Then, using (10), we obtain

$$\begin{aligned} \mathrm{P}[M \leq 1 - \eta/10] &\geq \frac{4\eta e^{-1.1}}{5} \\ &> \frac{\eta}{10}. \end{aligned}$$

Our claim follows clearly from the last inequality.

□

5 A PTAS for Dense Weighted Instances of MAX-BISECTION and MAX-CUT

In [AKK95] and [FV96] the following Theorem was proved.

Theorem 2. *0,1 dense MAX-CUT does have a PTAS.*

The following more general result can be proved in a similar way.

Theorem 3. *Assume the weights in each instance in the set I have mean equal to 1, and moreover assume that the weights are bounded above by an absolute constant. Then MAX-CUT and MAX-BISECTION on I both have PTASs.*

The crux of the methods of [AKK95] and [FV96] relies on so-called sampling lemmas which work when the dispersion of the weights is of comparable magnitude to that of their means. This is guaranteed by the assumptions of Theorem 3.

The following Theorem will be easily deduced from Theorem 3.

Theorem 4. *Let the family of representable d.f.'s $\mathcal{F}$ be dense (i.e. each $F \in \mathcal{F}$ has a finite support and rational probabilities and, moreover, $\mathcal{F}$ satisfies to the conditions of Theorem 1). Then* MAX-CUT*, and* MAX-BISECTION *both have PTASs when restricted to the instances corresponding to $\mathcal{F}$.*

PROOF We first need some notation. Given an underlying vertex set $V = V_n$ of size n and any subset $S \subseteq V$ we denote by $\delta(S)$ $(= \delta(V-S))$ the set of unordered pairs uv of vertices with $u \in S$, $v \in V - S$. Thus $\delta(S)$ is the cut defined by S in the complete graph with vertex set V_n.
For any instance I and any subset S of the corresponding graph, we denote by $val(I,S)$ the value of the cut defined by S:

$$val(I,S) := \Sigma_{e \in \delta(S)} w(e)$$

Here $w(e)$ is the weight of the edge e. If the instance is a graph, we write more simply $val(G,S)$ for $val(I,S)$. Hence we have

$$val(G,S) = |\delta(S) \cap E(G)|$$

where $E(G)$ denotes the edge set of G. Turning to the proof of Theorem 4, let $\mathcal{F}$ be dense, fix an $\epsilon > 0$ and let m_o be the minimum real number such that the inequality

$$s_o(m_o) \leq \frac{\epsilon}{2}$$

is satisfied. Here $s_o(.)$ is the function corresponding to $\mathcal{F}$ in condition (i) of Theorem 1. Now let I be an instance whose weight distribution coincides with some $F \in \mathcal{F}$). In order to approximate the maximum cut of I within $1-\epsilon$ we can proceed as follows.

- We replace by 0 all the weights exceeding m_o. Let I' denote the new entry.
- Since I' has bounded weights after standardisation, we can according to Theorem 3, find in polynomial time a cut $\delta(S)$ whose value $val(I',S)$ approximates that of a maximum cut of (I') within $1-\epsilon/2$, say.

Now, to see that $\delta(S)$ solves MAX-CUT within $1-\epsilon$ on the original instance I, observe that the total weight annihilated when going from I to I' does not exceed $\binom{n}{2}.\epsilon/2$. Thus, if $\mathrm{Opt}(I)$ is the maximum value of a cut of I, we have certainly

$$\begin{aligned} \frac{val(I',S)}{\mathrm{Opt}(I)} &\geq \frac{val(I',S)}{\mathrm{Opt}(I')}.\frac{\mathrm{Opt}(I')}{\mathrm{Opt}(I)}. \\ &\geq (1-\epsilon/2)^2 \geq 1-\epsilon \end{aligned}$$

where we have used in the last derivation the inequality $\mathrm{Opt}(I) \geq \frac{1}{2}\binom{n}{2}$. This concludes the proof for MAX-CUT. The proof for MAX-BISECTION is exactly the same. □

6 Hardness of MAX-BISECTION on a Non-Dense Set of Unweighted Instances

The strict converse of Theorem 4 which would state that MAX-BISECTION and MAX-CUT are MAX-SNP-hard on any non-dense set $\mathcal{F}$ does not hold. To see this, let us recall first the best time bound for dense MAX-CUT.

Theorem 5 [GGR96]. *For any fixed $d > 0$ and relative accuracy requirement ϵ, there is an algorithm which solves MAX-CUT on unweighted instances of density at least d in time at most*

$$C_1 n 2^{\frac{C_2}{d\epsilon^2}} \tag{12}$$

where C_1 and C_2 are absolute constants

Now let $\mathcal{F} = (F_i)_{i=1,2,\ldots}$ be a non-dense family of d.f's where F_i corresponds to the 0,1 instances with density d_i, say, and the sequence (d_i) tends to 0. (We always assume that the sequence (d_i) decreases.) Let $\frac{N_i}{D_i} = d_i$ be the shortest fraction expressing d_i. Then D_i divides $\binom{n_i}{2}$ where n_i is the smallest order of a graph on which F_i can be represented. Thus we certainly have $n_i \geq \sqrt{D_i}$. Assume $D_i \geq 2^{\frac{\lambda}{d_i}}$ for some fixed $\lambda > 0$ and all i. Then, the order n of any graph on which F_i is representable satisfies the inequality $n \geq 2^{\frac{\lambda}{2d_i}}$. Thus, according to (12), the time complexity $T(n)$ for computing MAX-CUT within $1-\epsilon$ on such a graph satisfies

$$T(n) \leq C_1 n 2^{\frac{C_2}{d_i \epsilon^2}} \leq C_1 n^{1+\frac{2C_2}{\lambda \epsilon^2}},$$

i.e. we have a PTAS for $\mathcal{F}$ with exponent $1 + \frac{2C_2}{\lambda\epsilon^2}$.

We thus need an upper bound for the denominators of the d_i's to obtain an inapproximability result in the 0,1 case and we will assume that the D_i's are bounded above by a polynomial function of the inverse of the density. We shall use a similar condition in the general weighted case (see Theorem 7). Besides these small denominators conditions, the proofs of the inapproximability results that we present require another condition which, in the 0,1 case, says roughly speaking, that the sequence of densities (d_i) does not decrease too fast (albeit it may decrease as fast as a double exponential). Let us now state these results.

Theorem 6 (MAX-SNP-hardness of MAX-BISECTION and MAX-CUT in the non-dense 0,1 case) *Assume that the sequence of rational densities (d_i) tends to 0 and, moreover, that it satisfies to the inequalities*

$$d_{i+1} \geq d_i^h, \; i = 1, 2, ... \tag{13}$$

where h is a positive constant. Assume moreover that the denominators D_i of the d_i satisfy

$$D_i \leq p(d_i^{-1}) \tag{14}$$

where $p(.)$ is a fixed polynomial.

Then, MAX-BISECTION and MAX-CUT are both MAX-SNP-hard on the set of 0,1 instances whose densities belong to (d_i).

Theorem 7 (MAX-SNP-hardness of MAX-BISECTION in the non-dense weighted case) *Let $\mathcal{F} = (F_i)_{i=1,2,...}$ be a non-dense family of representable d.f.'s each with mean 1, and, for each i, let D_i denote the smallest common denominator of the individual probabilities of the distribution F_i. Assume that there exist reals $\eta > 0$ and $h > 1$, and a sequence of numbers $(t_i)_{i=1,2,...}$ tending to infinity, s.t. the following three conditions hold for all $i \in \mathbb{N}$:*

$$t_i(1 - F_i(t_i)) \geq \eta, \tag{15}$$

$$D_i \leq p(t_i) \tag{16}$$

and

$$t_{i+1} \leq t_i^h. \tag{17}$$

Then, MAX-BISECTION is MAX-SNP-hard on the set of instances corresponding to $\mathcal{F}$.

Theorem 8 (MAX-SNP-hardness of MAX-CUT in the non-dense weighted case) *Let $\mathcal{F} = (F_i)_{i=1,2,...}$ be a non-dense family of representable d.f.'s each with mean 1, and assume that $\mathcal{F}$ fulfills the conditions of Theorem 7. Then, MAX-CUT is MAX-SNP-hard on the set of instances corresponding to $\mathcal{F}$.*

6.1 Proof of Theorem 6

The main step of the proof of theorem 6 is a reduction from the case of graphs with fixed average degree on which both MAX-BISECTION and MAX-CUT are Max-SNP hard as was proved by Papadimitriou and Yanakakis. In fact, and this will be important for us, Papadimitriou and Yanakakis prove that there is a set of graphs $\mathcal{G}_{PY}$ with the following properties:

- The optimum values of MAX-CUT and MAX-BISECTION coincide on each graph in $\mathcal{G}_{PY}$.

- MAX-CUT is MAX-SNP-hard on $\mathcal{G}_{PY}$.

(This implies of course that MAX-BISECTION is also MAX-SNP-hard on $\mathcal{G}_{PY}$.)

Moreover, we can assume that the valencies of the graphs in $\mathcal{G}_{PY}$ are bounded. Specifically, if δ denotes the average degree of a graph in $\mathcal{G}_{PY}$, we will assume the inequalities $1 \leq \delta \leq D$ where D is a fixed number.

Definition(Asymptotically equal sides condition): Let $\mathcal{G} = \cup_{n \in L} \mathcal{G}_n$ be a family of graphs where the graphs in $\mathcal{G}_n$ have n vertices. We say that $\mathcal{G}$ satisfies to the asymptotically equal sides condition, AES condition for short, iff each $G \in \mathcal{G}_n$ has an optimum cut A, B whose sides satisfy $||A| - |B|| = o(n)$.

We need several lemmas. We denote by $\mathcal{G}(n, d)$ the set of graphs with n vertices and average degree d.

Lemma 3 *Let an integer h and a family of graphs $\mathcal{G} = \{G_i : i \in J\}$ be given. For each i, let H_i denote the join of G_i with an independent set of size h (i.e. we make h replicas of each vertex of G_i and each edge of G_i gives a complete bipartite graph between the two corresponding sets of replicas). Let $\mathcal{H} = \{H_i : i \in J\}$ Then, the problems of approximating MAX-BISECTION in $\mathcal{G}$ and $\mathcal{H}$ are mutually L-reducible one to the other.*

PROOF Let A be an algorithm for MAX-BISECTION with approximation ratio ρ on $\mathcal{H}$. For each vertex $x \in V(G_i)$, its h replicas are equivalent. It follows easily that, given an equi-cut (A, B) and two vertices $x, y \in V(G_i)$, which both have replicas in A and B, it is always possible either to move a copy of x from A to B and a copy of y from B to A or to move a copy of x from B to A and a copy of y from A to B without lessening the value of the bisection. We can thus assume that, except for at most one exception, the replicas of each fixed vertex of G_i all go to the same side of the equi-cut $A(H_i)$. We can moreover assume that this exceptional vertex, if any, has minimum degree in G_i. Thus A translates with minor modifications into an algorithm for MAX-BISECTION on $\mathcal{G}$ with approximation ratio $(1 - O(1/n))\rho$ on instances of size n. This proves the L-reducibility in one direction. The other direction is straightforward.

□

Lemma 4 *Let Δ be a sufficiently large real number and let $\Delta' > \Delta$. MAX-BISECTION and MAX-CUT are MAX-SNP-hard on any set of graphs*

$$\mathcal{H} = \cup_{n \in \mathbb{N}} \mathcal{G}(n, d_n)$$

where the d_n's satisfy $\Delta \leq d_n \leq \Delta'$ for each $n \in \mathbb{N}$.

PROOF Let G be a graph with n vertices and average degree δ, $1 \leq \delta \leq D$. Consider a fixed sequence (d_n) and assume that it satisfies to the condition of the lemma. Put $h = \lfloor \frac{d_n}{\delta} \rfloor$.

Denote by G' the join of G with an independent set of size h. G' has average degree greater than $d_n - \delta$. Then, by adding less than $(1/2)\delta n$ arbitrary edges to G', we obtain a graph G" with average degree d_n (which belongs to $\mathcal{H}$). Fix an arbitrarily small positive ϵ and let $(V_1", V_2")$ be a partition of $V(G")$ which approximates MAX-BISECTION within $1 - \epsilon$. Then, $(V_1", V_2")$ approximates MAX-BISECTION within $1-\epsilon-\delta/(2\Delta)$ on G'. Using Lemma 3, we deduce immediately from $(V_1", V_2")$ a partition (V_1, V_2) which approximates MAX-BISECTION on G within $1 - \epsilon - \frac{\delta}{\Delta}$, say. This clearly contradicts the MAX-SNP-hardness of MAX-BISECTION if ϵ is sufficiently small and Δ sufficiently large. Now, by restricting G to belong to $\mathcal{G}_{PY}$ we see that $\mathcal{H}$ contains a subset $\mathcal{H}'$, say, which is MAX-SNP-hard for MAX-BISECTION and with the property that MAX-CUT and MAX-BISECTION coincide within $1 + o(1)$ on $\mathcal{H}'$. This implies clearly that MAX-CUT is also MAX-SNP-hard on $\mathcal{H}'$.

□

Lemma 5 *Let Δ be a sufficiently large real number and let $\Delta' > \Delta$. Assume that the sequence $(n_k)_{k=1,\dots}$ satisfies for any sufficiently large k to the inequality*

$$n_{k+1} \leq n_k^h \tag{18}$$

where h is a fixed number greater than 1. Then, there exists a set $\mathcal{K}$ of graphs MAX-SNP-hard for bisection and with the following properties

- *The average degree of each graph in $\mathcal{K}$ belongs to the interval $[\Delta, \Delta']$*
- *The vertex set sizes belong to (n_k),*
- *The graphs in $\mathcal{K}$ satisfy to the AES condition.*

Note that the last assertion of the Lemma implies that the set $\mathcal{K}$ is also MAX-SNP-hard for MAX-CUT.

PROOF Let $\mathcal{K}$ be the set of graphs in $\mathcal{H}'$, (defined in the proof of Lemma 4) whose vertex set sizes belong to (n_k). Assume for a contradiction that for any $\epsilon \geq 0$, there exists an integer k such that 0,1 MAX-BISECTION is $(1 - \epsilon)$-approximable in time $poly(n_k)$ on $\mathcal{K}$. by some algorithm A. Set for each n, $m = m(n) = \min\{n_j : n_j \geq \frac{n}{\epsilon}\} = n_q$, say. We have $m \leq n^{h+1}$ for sufficiently large n. Let $\lambda = \lfloor \frac{m}{n} \rfloor$ and associate to each instance I of size n the join J of λ copies of I. Eventually add isolated vertices to obtain an instance J' of order m. Clearly, an approximate solution of J' is also an approximate solution of J and by Lemma 3, we can deduce in polynomial time from an approximate solution of J' an approximate solution of I with the same approximation ratio. Thus the algorithm A can be used with trivial modifications to approximate MAX-BISECTION for any instance of size n in $\mathcal{H}'$ in time n^{kh}. This contradicts the MAX-SNP-hardness of MAX-BISECTION in $\mathcal{H}'$. $\mathcal{K}$ satisfies to the AES condition simply because because $\mathcal{H}'$ does. □

We are now well prepared for the proof of Theorem 6.

Proof of Theorem 6

Let the sequence of densities $\mathcal{D} = (d_i)$ satisfy to the conditions of the Theorem. Fix an arbitrary small $\epsilon > 0$ and define from $\mathcal{D}$ a new family $\mathcal{D}'$ where for each i, d_i is replaced by a δ_i satisfying

$$(1 - \epsilon)d_i \leq \delta_i \leq d_i \tag{19}$$

and having a shortest fractional expression, say $\delta_i = \frac{P_i}{Q_i}$, with $\lceil \frac{1}{\epsilon} \rceil \leq P_i \leq \lceil \frac{2}{\epsilon} \rceil$. Let us show first that MAX-BISECTION is hard to approximate on $\mathcal{D}'$. which will clearly imply that it is hard to approximate on $\mathcal{D}$. Because of Lemma 5 we need only an infinite sequence of sizes (n_k) such that, for each k, the average degrees $\overline{\delta}_k$ of the graphs on n_k vertices and with density δ_k in $\mathcal{D}'$ belong to some fixed interval $[\Delta, \Delta']$ with Δ sufficiently large. We shall take $\Delta' = \Delta.\lceil \frac{2}{\epsilon} \rceil$. For a graph on n_k vertices with density δ_k we have

$$\overline{\delta}_k = (n_k - 1)\delta_k = (n_k - 1).\frac{P_k}{Q_k}$$

Thus, if we choose $n_k = \Delta Q_k + 1$, we get $\overline{\delta}_k = \Delta P_k$ implying $\Delta \leq \overline{\delta}_k \leq \Delta'$ as desired. It remains to observe that (13) implies the inequality

$$\delta_{i+1} \geq \delta_i^{h+1}$$

for all sufficiently large i.

□

Let $\mathcal{G}_\mathcal{D}$ stand for the graphs whose densities belong to the set $\mathcal{D}$ and let $\mathcal{H}" = \mathcal{G}_\mathcal{D} \cap \mathcal{G}_{PY}$. Of course, we can carry over the above proof starting with the subset $\mathcal{H}" \subseteq \mathcal{G}_\mathcal{D}$ and ending with the conclusion that this set is MAX-SNP-hard for MAX-BISECTION (and MAX-CUT). We shall use the set $\mathcal{H}"$ in the proof of Theorem 8. For ease of reference, let us restate its properties in a separate Lemma.

Lemma 6 *Assume that the sequence of rational positive numbers (d_i) tends to 0 and, moreover, that it satisfies to the inequalities $d_{i+1} \geq d_i^h$, $i = 1, 2, ...$ where h is a positive constant. Assume moreover that the denominators D_i of the d_i satisfy*

$$D_i \leq p(d_i^{-1}) \tag{20}$$

where $p(.)$ is a fixed polynomial. There is then a set $\mathcal{H}"$ of graphs whose densities belong to the sequence (d_i) which satisfies to the AES condition and which is MAX-SNP-hard for MAX-BISECTION (and MAX-CUT)

Proof □

7 Proof of Theorem 7

The following Lemma asserts broadly speaking that putting random weights with mean 1 on the edges of a (not too sparse) graph G does not change significantly the maximum value of a cut of G.

Lemma 7 (Averaging Lemma) *Let (G_n) be a sequence of graphs where G_n has n vertices and $m = m(n)$ edges and $n = o(m)$ and let (F_n) be a dense sequence of distributions. Assume that for each n the edges of G_n are given random non-negative weights picked from F_n. Let G'_n denote this weighted graph.*
The quantity

$$\frac{1}{m} \max_S |val(G'_n, S) - val(G_n, S)|,$$

where S ranges over all subsets of $V(G)$, tends to 0 in probability when $n \to \infty$.

PROOF We first get rid of the extreme values of F. Define $\theta = \theta(\epsilon)$ by

$$\int_\theta^\infty s dF(s) = \frac{\epsilon^2}{4}, \tag{21}$$

and note for future use that 5 implies that there is a function $f(.)$ such that the inequality

$$\theta \leq f(\epsilon)$$

holds for every n. Then the expectation of the total weight of the edges with weights $\geq \theta$ is equal to $\frac{m\epsilon^2}{4}$. Thus by Markov inequality, this weight does not exceed $\frac{m\epsilon}{2}$, with probability at least $1-\frac{\epsilon}{2}$. This implies clearly

$$\Pr[\max_S |val(G'_n, S) - val(G''_n, S)| \leq \epsilon m/2] \geq 1 - \epsilon/2 \ . \tag{22}$$

where G_n" denotes the subgraph of G'_n spanned by the edges with weights less than θ.
To proceed now with G_n", call F' the distribution obtained from F by replacing the values $\geq \theta_n$ by 0. Thus the weights of the edges of G_n" are distributed according to F'. Let us denote by E' the expectation of F' : $E' = 1 - \epsilon^2/4$.
Let us fix a cut $\delta(S)$ with $val(G_n, S) = |\delta(S) \cap E(G_n)| = q$, say, Now, since $val(G_n", S)$ is the sum of q independent r.v.'s with the common distribution F' each bounded above by θ, the Chernoff-Hoeffding bound (see [HOE63]) gives

$$\begin{aligned} \Pr[|val(G_n", S) - qE'_n| \geq \frac{\epsilon m}{3}] &\leq 2\exp\left(-\frac{\epsilon^2 m^2}{9\theta^2 q}\right) \\ &\leq 2\exp\left(-\frac{\epsilon^2 m}{9\theta^2}\right). \end{aligned}$$

We have $val(G_n, S) = q$ and $E' \geq 1 - \epsilon/3$. The preceding inequality gives, with (22),

$$\Pr[|val(G_n", S) - val(G_n, S)| \geq \frac{2\epsilon m}{3}] \leq 2\exp\left(-\frac{\epsilon^2 m}{9\theta^2}\right).$$

Since the total number of cuts is bounded above by 2^n, we obtain

$$\begin{aligned} \Pr[max_S|val(G_n", S) - val(G_n, S)| \geq \frac{2\epsilon m}{3}] &\leq 2^{n+1}\exp\left(-\frac{\epsilon^2 m}{9\theta^2}\right) \\ &\leq \frac{\epsilon}{2} \end{aligned}$$

for sufficiently large n by our assumption on $m = m(n)$ and the inequality $\theta \leq f(\epsilon)$. Now inequality (22) implies

$$\Pr[\max_S |val(G'_n, S) - val(G_n, S)| \leq \epsilon m] \geq 1 - \epsilon,$$

and the theorem, since ϵ is arbitrary.

$\square$

7.1 End of the Proof of Theorem 7

Our strategy for obtaining an hardness result for the general (weighted) case of MAX-BISECTION is to reduce it to the 0,1 case.

Theorem 9 *Assume that $\mathcal{F} = (F_i)$ satisfies to the conditions of Theorem 7 with parameters η and h. Then, approximating MAX-BISECTION on $\mathcal{F}$ L-reduces to approximating MAX-BISECTION on a non-dense set of 0,1 instances.*

PROOF Let the sequences (t_i) and (D_i) satisfy to the conditions of Theorem 7 and let $\mu_i = 1 - F(t_i)$. We have $t_i \geq \frac{\eta}{\mu_i}$. Thus (16) implies

$$D_i \leq q(\mu_i^{-1}) \tag{23}$$

where q is another polynomial. Also, it is not hard to show that (17) implies the existence of a subsequence $(j(i))$ of the natural integers with

$$\mu_{j(i+1)} \geq (\mu_{j(i)})^{h+1}.$$

We can thus assume by renaming that we have

$$\mu_{i+1} \geq \mu_i^{h+1} \tag{24}$$

for every $i \in \mathbb{N}$.

We proceed now to transform each instance in an instance with only two distinct weights with the help of the Averaging Lemma.

For a fixed i, set $F \equiv F_i,\ t = t_i$ and define

$$\alpha = \alpha_i = \frac{1}{1 - F(t)} \int_t^{\infty} sdF(s)$$

and

$$\beta = \beta_i = \frac{1}{F(t)} \int_0^t sdF(s).$$

For $n = 2\lambda D_i,\ \lambda \in \mathbb{N}$, set $m = \mu_i \binom{n}{2}$, (m is an integer because F_i is representable on a graph with n vertices), and use k to index the $\binom{\binom{n}{2}}{m}$ distinct subgraphs G_k of K_n having m edges. We define for each k a partial instance J_k by giving to the edges of G_k random weights empirically distributed according to the d.f.

$$G(s) = \frac{F(s) - F(t)}{1 - F(t)},\ s \geq t.$$

We define also a partial instance L_k by putting on the edges in the set $K_n \backslash G_k$ random weights on $[0, t]$ empirically distributed according to the d.f. $H(s) = F(s)/F(t),\ s \leq t$. We denote by I_k the instance obtained by sticking together J_k and L_k. Clearly, by the choice of $G(.)$ and $H(.)$, the empirical distribution of the weights in I_k coincides with F.

Assume that one can find in polynomial time a bisection $\delta(S_o)$ with value $val(I_k, S_o) \geq (1 - \epsilon)Opt(I_k)$. Let I'_k denote the instance obtained by replacing the weights in J_k (resp. in L_k) by their mean α, (resp. β).

We can apply the averaging Lemma separately to J_k and L_k, which implies that the maximum value of a bisection in I'_k does not differ from $val(I'_k, S_o)$ by more than a $1 - o(1)$ factor.

Now let I_k" denote the instance obtained from I'_k by substracting β from each weight. (Thus I_k" has weights all equal to $\alpha - \beta$ on the edges of G_k and zero weights elsewhere.) For any bisection $\delta(S)$ we have clearly

$$val(I'_k, S) = val(I"_k, S) + \frac{\beta n(n-1)}{4}\ . \tag{25}$$

Note that we have $\alpha(1 - F(t) \geq \eta$ and since we have

$$\alpha(1 - F(t)) + \beta F(t) = 1,$$

while $F(t) = F_i(t_i)$ tends to 1 with i, we deduce that β has an upper bound strictly smaller than 1. Since the maximum value of a bisection of I_k is at least $n(n-1)/4$, this implies for $S = S_o$ that the ratio

$$\frac{val(I"_k, S_o)}{val(I'_k, S_o)}$$

is bounded below by a strictly positive constant so that $\delta(S_o)$ is also an approximate solution for MAX-BISECTION on $I"_k$. Thus, approximating MAX-BISECTION on $\mathcal{F}$ enables us to approximate 0,1 MAX-BISECTION on the graphs with densities μ_i and orders D_i, under conditions (23) and (24.) This clearly contradicts Theorem 6. □

8 Proof of Theorem 8

In order to prove Theorem 8, we proceed as in the proof of Theorem 7. However, equality (25) is no longer true since we can by no means guarantee that the maximum cut is a bisection. To cure this, we restrict the graphs G_k to belong to the set $\mathcal{H}$" as defined in Lemma 6 and corresponding to the $\mu_i's$ and $D_i's$ of Theorem 7 and Theorem 8, which satisfies to the AES condition. For this restricted set, the maximum cut of the whole instance I_k" is also (almost) a bisection and, by using the same reasoning as for the proof of Theorem 7, we deduce that finding an approximately maximum cut of I_k" allows one to find an approximately maximum bisection of J_k which is not possible by Theorem 6. Theorem 8 follows.

□

9 Summary and Conclusions

With the aim of separating as sharply as possible the approximable from the inapproximable families of weighted instances of MAX-CUT, we have introduced a notion of dense families of instances or, more precisely, a notion of dense families of weight distributions. We have shown that the corresponding families of instances have the (intended) approximability property for MAX-CUT.

In the other direction, we have shown inapproximability only when the densities in the set of instances do not decrease too fast, and we believe that this condition is not necessary. This is our first question.

A second question is: Does our density definition capture the approximability of all MAX-SNP-hard problems in the weighted case? We know by [AKK95] that all these problems are approximable in the dense *unweighted* case.

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References

[AFK96] S. Arora, A.M. Frieze and H. Kaplan, A New Rounding Procedure for the Assignment Problem with Applications to Dense Graph Arrangements, *Proc. 37th IEEE FOCS (1996) pp.21-30*

[AKK95] S. Arora, D. Karger, and M. Karpinski, Polynomial Time Approximation Schemes for Dense Instances of NP-Hard Problems, *Proc. 27th ACM STOC (1995) pp. 284-293*

[CS97] M. Cesati and L. Trevisan, *On the Efficiency of Polynomial Time Approximation Schemes*, To appear in *Information Processing Letters, 1997*

[TR97] P. Crescensi, R. Silvestri and L. Trevisan, To Weight or not to Weight: Where is the Question? *Proc. IEEE, ISTCS'96 (1996)*

[Fe] W. Feller, *An Introduction to Probability Theory and its Applications,* Vol 2. Wiley 1963

[FV96] W. Fernandez de la Vega, MAX-CUT has a Randomized Approximation Scheme in Dense Graphs, *Random Structures and Algorithms, Vol. 8. No.3 (1996) pp.187-198*

[AK96] A.M. Frieze and R. Kannan, The Regularity Lemma and Approximation Schemes for Dense Problems, *Proc. 37th FOCS (1996) 12-20.*

[AK97] A.M. Frieze and R. Kannan, Quick Approximation to Matrices and Applications, *Manuscript (1997)*

[GJ79] M.R. Garey and D.S. Johnson, *Computers and Intractability, A Guide to the theory of NP-Completeness.* Freeman, San Francisco CA,1979.

[GGR96] O. Goldreich, S. Goldwasser and D. Ron, Property Testing and its Connection to Learning and Approximation, *Proc. 37th FOCS (1996)*

[HOE63] W.J. Hoeffding, Probability Inequalities for Sums of Random Variables, *J. Amer. Stat. Assoc. , Vol 58, 713-721, (19630)*

[PY91] C.H. Papadimitriou and M. Yannakakis, Optimization, Approximation, and Complexity Classes, *Journal of Computer and System Sciences, Vol 43, 425-440 (1991)*